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A New Explanation of Relation between Matrices

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Abstract

It is known that if two matrices are of same size, there may be the equivalent, similar or congruent relations between matrices. This paper has mainly launches the research about relations between matrices of the same size, different size by matrix operations. Such as there is an association between matrix and its adjoint matrix, adjoint matrix is linear combination of power of A. Using a mathematical formula to unify the three equivalence relations.

Keywords: *matrix relations classification matrix operations*

1. Introduction

About the relationship between matrices , most of the literature focuses on the equivalence relation , similar relation, congruent relation , for example , [1]-[6] has discussed the three important relations in the matrix : equivalence , similarity , congruence , and analyzed the difference and connection between them . [7] has induced and analyzed the differences between matrix , the number of operations , the internal relations of matrix operations and applications of matrix operations in the determinant of a matrix , rank of matrix , the relation between matrices , elementary transformations . Because three relations are equivalence relations and matrices can be classified basing on any kind of equivalence relation. Three relations mentioned above can be defined by a matrix multiplication operation as follow.

Let A , B be matrices of the same type , A=PBQ , where P , Q are invertible matrices , then A is equivalent to B, Further , if A , B are square matrices of the same order , Q is the inverse matrix of P , then A is similar to B . If A, B are real symmetric matrices of the same order, Q is the transpose of P, then A is congruent to B.

Since three relations above depends on the multiplication operation of matrix, then whether other relations between matrices can also be given by matrix operations, whether there is an association between matrix and its transpose matrix, adjoint matrix .what relation between two matrices of different row (column) This paper has mainly launches research about relations between matrices of the same size, different size by matrix operations. Such as the relation between two matrices of same column.

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

2. Preparation of Knowledge

Definition 1: A matrix A is said to be equivalent to a matrix B if there exist invertible matrix P, Q, such that PAQ=B.

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Definition 2: A matrix A is said to be similar to a matrix B if there exists an invertible matrix P such that $P^{-1}AP = B$.

Definition 3: A matrix A is said to be congruent to a matrix B if there exists an invertible matrix P such that $P^{T}AP = B$.

Definition 4: A relation is said to be an equivalence relation if the relation satisfy reflexivity, symmetricity and transitivity.

It is easy to see that the equivalence, the similarity and the congruence of matrices are three equivalence relations. Reflexivity, symmetricity and transitivity, by these three properties, all matrices are classified into classe. Every class has a representative elemen, all matrices in the same class are equivalent to each other, and matrices in different class are not equivalent.

3. Main Results

3.1. Relation between Matrix with its Transpose and Adjoint Matrix.

When $m \neq n$, let $A \in F^{m\times n}$, supposed $m \leq n$, k is nonzero number, then

$$(kA)^{T} = kA^{T}, R\left(A^{T}\right) = R(A), R(kA) = R(A). \tag{1}$$

When $m = n$, Let $A \in F^{n\times n}$, A^{T} is the transpose matrix of A , and A^{*} is the adjoint matrix of A, E_n is an identity matrix of order n , then there is a very important equation about A and A^{*}, such that $AA^{*} = A^{*}A = |A|E_n$.

Some property of the transpose matrix or adjoint matrix A^{*} of A may depend on the original matrix A. Such as, let k a nonzero number.

$$\left|A^{T}\right| = |A|, |kA| = k^{n}|A| \tag{2}$$

$$\left|A^{*}\right| = |A|^{n-1}(n \geq 2), R\left(A^{*}\right) = \begin{cases} n, R(A) = n \\ 1, R(A) = n-1 \\ 0, R(A) < n-1 \end{cases}. \tag{3}$$

Theorem 1: Let A be a square matrix of order n, then adjoint matrix A^{*} of A is linear combination of power of A.

In particular, when A is invertible, then

$$A^{*} = -\frac{|A|}{a_0}\left(A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1E_n\right) \tag{4}$$

Where $f(\lambda) = |\lambda E_n - A| = \lambda^{n} + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$ is the eigenpolynomial of A.

Proof: Let A be a square matrix of order n, the following discussion according to the rank of matrix A from three aspects.

When $R(A) = n$, A is a matrix of full rank, and A is an invertible matrix.

$$f(\lambda) = |\lambda E - A| = \lambda^{n} + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 \tag{5}$$

It is the eigenpolynomial of *A*. By Hamilton-Cayley theorem $f(A) = 0$, that is to say

$$A\left(A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1E\right) + a_0E = 0 \tag{6}$$

Because A is invertible, then $f(0)=\left|-A\right|=a_0\neq 0$, and

$$A\left(-\frac{1}{a_0}A^{n-1}+-\frac{a_{n-1}}{a_0}A^{n-2}-\cdots-\frac{a_1}{a_0}E\right)=E \quad (7)$$

Then

$$A^{-1}=-\frac{1}{a_0}\left(A^{n-1}+a_{n-1}A^{n-2}+\cdots+a_1E\right) \quad (8)$$

and $A^{-1}=\frac{1}{\left|A\right|}A^*$,

That is

$$A^*=-\frac{\left|A\right|}{a_0}\left(A^{n-1}+a_{n-1}A^{n-2}+\cdots+a_1E\right)=-\frac{\left|A\right|}{a_0}(1,a_{n-1},\cdots,a_1,a_0)\begin{pmatrix}A^{n-1}\\A^{n-2}\\\cdots\\A\\E\\0\end{pmatrix} \quad (9)$$

So A^* is the polynomial of A.

When $R(A)=n-1$, let $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$ be a basis of linear space V, A,A^* are the matrix of linear transformation A,A^* with respect to the basis $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$.

Because $f(A)=A^n+a_{n-1}A^{n-1}+\cdots+a_1A$ is an annihilator polynomial of A, then there exist B_1 , polynomial of A, such that $B_1\neq 0, AB_1\neq 0$.

One side, because $R(A)=n-1$, so let $A\varepsilon_1,A\varepsilon_2,\cdots,A\varepsilon_{n-1}$ be independent vectors. $AV=L(A\varepsilon_1,A\varepsilon_2,\cdots,A\varepsilon_{n-1})$ .But $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$ are independen, then there is any vector ε_i which can not be linear represented by $A\varepsilon_1,A\varepsilon_2,\cdots,A\varepsilon_{n-1}$. Suppose it is ε_1 , let B_1 be linear transformation of corresponding to matrix B_1 , then $B_1\varepsilon_1$ is not zero , but $AB_1\varepsilon_1=0$, $B_1\varepsilon_1$ is in $A^{-1}=(0)$.

On the other hand, $AA^*\varepsilon_1=0$, then $A^*\varepsilon_1$ is also in $A^{-1}=(0)$, so there is number k, such that $A^*\varepsilon_1=kB_1\varepsilon_1$. Because The role of linear transformation kB_1,A^* in $A\varepsilon_1,A\varepsilon_2,\cdots,A\varepsilon_{n-1}$ are all zero . And $A\varepsilon_1,A\varepsilon_2,\cdots,A\varepsilon_{n-1}$ are independent, $A^*=kB_1$. So A^* is the polynomial of A.

When $R(A)<n$, $R\left(A^*\right)=0$, that is $A^*=0$, So A^* is a zero polynomial and the polynomial of A .

Example 1: For $A=\begin{pmatrix}1&1\\-2&3\end{pmatrix}$.

$\left|A\right|=5\neq 0$, then A is invertible.

One hand,

$$A^{*} = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}. \tag{10}$$

On the other hand, the eigenpolynomial of *A* of order 2 is

$$f(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 1 & -1 \\ 2 & \lambda - 3 \end{vmatrix} = \lambda^{2} - 4\lambda + 5 \tag{11}$$

Where $a_1 = -4, a_0 = 5$.

Then

$$A^{*} = -\frac{|A|}{a_0}\left(A^{2-1} + a_1 E_2\right) = -\frac{5}{5}(A - 4E_2) = \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}. \tag{12}$$

Moreover some property of the transpose matrix or adjoint matrix of A may inherit from the original matrix A, for example symmetry, invertible, the positive definiteness, orthogonality.

3.2. Relation between two Matrices by their Size

Row number, column number are two important index of matrices , Many of the properties of matrix relate to their indexes , such as the rank of a matrix , standard form of matrix etc .

3.2.1. When two Matrices have the Same Size of m Row and n Column. Suppose $A, B \in F^{m\times n}$. $m \neq n$, suppose that $m \leq n$, A , B have the same size , we can only consider the equivalent matrices . Let A, B be matrices of the same size, we can think about whether two matrices are equivalent.

Proposition 1: Two $m \times n$ matrices are equivalent if and only if they have the same rank.

Based on the equivalent relation, matrices can be divided into class m+1= min {m, n}+1 ,

$$O, \begin{pmatrix} E_1 & \\ & O_{(m-1)\times(n-1)} \end{pmatrix}, \begin{pmatrix} E_2 & \\ & O_{(m-2)\times(n-2)} \end{pmatrix}, \ldots, (E_m, O_{n-m}) \tag{13}$$

as the representative element .

m = n, A and B are square matrices , We can consider the similar or congruent relations .

When two matrices are similar.

Proposition 2: A is similar to B if and only if their eigenmatrices $\lambda E_n - A$ and $\lambda E_n - B$ are equivalent.

Proposition 3: Let A and B be $n \times n$ matrices over a number field. The following conditions are equivalent.

A and B are similar.

A and B have the same determinant divisors.

A and B have the same invariant factors.

A and B have the same elementary divisors.

Proposition 4: Every matrix A is similar to a block diagonal matrix over C which consists of Jordan blocks down the diagonal. Moreover, these blocks are uniquely determined by A up to order.

Proposition 5: Let $A, B \in C^{m \times n}$, A is similar to B if and only if they have the same Jordan canonical form.

A is similar to B if and only if heir elementary divisors collection $(\lambda - \lambda_1)^{n_1}, \ldots, (\lambda - \lambda_k)^{n_k}$ are the same completely, where $\lambda_i, n_i, i = 1, \ldots,$ k, but $\lambda_i \in F$ has infinite, so based on the similar relation, matrices in number field can be divided into infinite class .

When two matrices are congruent.

Proposition 6: Two real symmetric matrices are congruent over R if and only if two of the followings are satisfied:

They have the same rank;

They have the same positive index;

They have the same negative index;

They have the same signature.

Based on the congruent relation, symmetric matrices in Real number field can be divided into1+2+3+…+ (n+1)= $\frac{(n+1)(n+2)}{2}$ class .

$$O, \begin{pmatrix} 1 & \\ & O_{(n-1)\times(n-1)} \end{pmatrix}, \begin{pmatrix} -1 & \\ & O_{(n-1)\times(n-1)} \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & & O_{(n-2)\times(n-2)} \end{pmatrix}$$

$$\begin{pmatrix} 1 & & \\ & -1 & \\ & & O_{(n-2)\times(n-2)} \end{pmatrix}, \begin{pmatrix} -1 & & \\ & -1 & \\ & & O_{(n-2)\times(n-2)} \end{pmatrix}, \ldots, \begin{pmatrix} E_p & & \\ & E_{r-p} & \\ & & O_{(n-r)\times(n-r)} \end{pmatrix} \tag{14}$$

as the representative element .

Three relations mentioned above can be defined by a matrix multiplication operation.

Theorem 2: Let A, B be matrices of the same type, A=PBQ , where P , Q are invertible matrices , then A is equivalent to B , further , if A, B are square matrices of the same order , Q is the inverse matrix of P , then A is similar to B . If A, B are real symmetric matrices of the same order, Q is the transpose of P , then A is congruent to B .

Example 2: For $A = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}$.

Because

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix} \tag{15}$$

Then A is equivalent to B, and the product decomposition of the equation is not unique.

Their eigenmatrices are not equivalent if and only if A is not similar to B

$$\lambda E - A = \begin{pmatrix} \lambda - 1 & 1 \\ -2 & \lambda - 2 \end{pmatrix}, \lambda E - B = \begin{pmatrix} \lambda + 1 & 0 \\ 0 & \lambda - 4 \end{pmatrix} \tag{16}$$

B is a real symmetric matrix and A isn’t symmetric, so A is not congruent to B.

The matrix classification

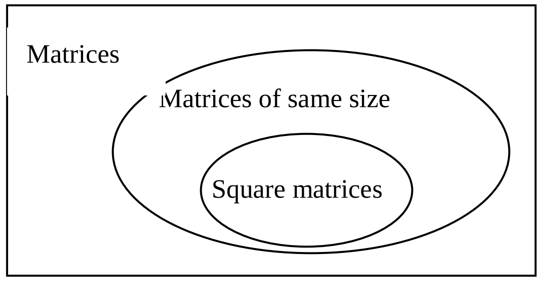

Figure 1. Venn Diagram of the Set of Matrices

The matrix will be divided into two classes in Figure 1: the same size of matrix and different size matrix . Teaching material mainly research the same size of matrices.

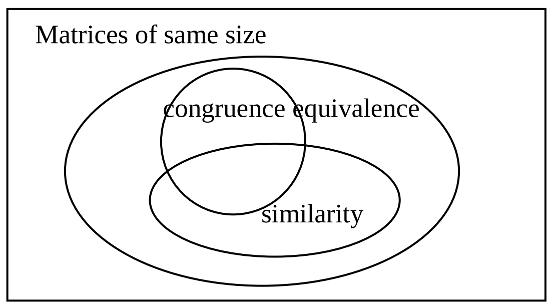

Figure 2. Venn Diagram of the Set of Matrices of Same Size

In the same size of matrices in Figure 2 can be divided into the matrix of equivalence , similarity , congruence , these three kinds of relations are equivalence relations, satisfying reflexivity , symmetricity and transitivity .

In particular, if the two square matrices of same order are similar, then they must be equivalent. In the same way, if the two square matrices of same order are congruent, then they are equivalent, but if two matrix are equivalent, but they are not necessarily similar or congruent.

3.2.2. When there is one index and only one of the same. Let $A \in F^{s\times n}, B \in F^{t\times n}, s \neq t$, suppose $s \leq t$, it exists a matrix C, such that $\begin{pmatrix} A \\ C \end{pmatrix} \in F^{t\times n}$, $\begin{pmatrix} A \\ C \end{pmatrix}$ and B have the same size.

Proposition 7: If $R\begin{pmatrix} A \\ C \end{pmatrix} = R(B) = n$, $\begin{pmatrix} A \\ C \end{pmatrix}$ and B are the full column rank , then $\begin{pmatrix} E_n \\ O_{(t-n)\times n} \end{pmatrix}$ can be written as $P\begin{pmatrix} A \\ C \end{pmatrix} = QB = \begin{pmatrix} E_n \\ O_{(t-n)\times n} \end{pmatrix}$, where P , Q are invertible matrices . it contains that $\begin{pmatrix} A \\ C \end{pmatrix} = P^{-1}QB$.

Example 3: Let $A = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}$.

Because $R\begin{pmatrix} A \\ C \end{pmatrix} = R(B) = 2$, then $\begin{pmatrix} A \\ C \end{pmatrix}$ and B are the full column rank , suppose

$C = (a, b)$ such that $\begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 2 \\ a & b \end{pmatrix}$,

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -(a+b) & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -a & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & 2 \\ a & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{17}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{18}$$

Then $\begin{pmatrix} E_2 \\ O_{1\times 2} \end{pmatrix}$ may be written as $P\begin{pmatrix} A \\ C \end{pmatrix} = QB = \begin{pmatrix} E_2 \\ O_{1\times 2} \end{pmatrix}$, which

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -(a+b) & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -a & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ -\frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2}(b-a) & -\frac{1}{4}(a+b) & 1 \end{pmatrix} \tag{19}$$

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix} \tag{20}$$

It contains that

$$\begin{pmatrix} 1 & -1 \\ 2 & 2 \\ a & b \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ -\frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2}(b-a) & -\frac{1}{4}(a+b) & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 1 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 1 \end{pmatrix} \tag{21}$$

If $R\begin{pmatrix} A \\ C \end{pmatrix} = R(B) = t$, $\begin{pmatrix} A \\ C \end{pmatrix}$ and B are the full row rank , then $\left(E_t, O_{t\times(n-t)}\right)$ may be written as $\begin{pmatrix} A \\ C \end{pmatrix}$ P =BQ= $\left(E_t, O_{t\times(n-t)}\right)$.

If $R(B) = R\begin{pmatrix} B \\ A \end{pmatrix} = t$, then $R(B^T) = R\left(B^T, A^T\right) = t$, then matrix equation $B^T X^T = A^T$ has the unique solution, therefore matrix equation $XB = A$ has the unique solution.

Proposition 8:

Let G be a full column matrix of $m \times n$, H be a full row matrix of $p \times q$, then for any $n \times p$ matrix A satisfies $R(GA) = R(A) = R(AH)$.

Let A be a $m \times n$ matrix of rank r, then there are P full column matrix of $m \times r$, Q be a full row matrix of $r \times n$ such that A=PQ.

Let $A \in F^{m\times p}, B \in F^{m\times q}, p \neq q$, it is no longer to be discussed here.

3.2.3. When Row Index, Column Index of two Matrices are Cross Correlation Equal

Let $A \in F^{s\times n}, B \in F^{n\times t}$, When $s \neq t$, suppose $s \leq t$, then $\begin{pmatrix} A \\ C \end{pmatrix}$ and B^T have the same size. Here it is no longer to be discussed.

When $s = t$, suppose $s \leq n$, then A and B^T have the same size, the products AB and BA always make sense and these products are square of sizes $s\times s, n\times n$ respectively. But generally equality AB=BA does not hold.

Proposition 9:

AB and BA have the same trace: *tr* (AB) = *tr* (BA).

$E_s - AB$ and $E_n - BA$ have the same invertibility : E_s -AB is invertible if and only if E_n -BA is invertible .

$$\lambda^{n-s}\left|\lambda E_s - AB\right| = \left|\lambda E_n - BA\right| . \quad (22)$$

Example 4: For $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 2 \\ 1 & -1 & -4 \end{pmatrix}$.

$$AB = \begin{pmatrix} 2 & 1 & -2 \\ 3 & 3 & 0 \\ 0 & 3 & 6 \end{pmatrix}, BA = \begin{pmatrix} 7 & 1 \\ -5 & 4 \end{pmatrix} . \quad (23)$$

$$tr\,(AB) = 11 = tr\,(BA). \quad (24)$$

$$E_3 - AB = \begin{pmatrix} -1 & -1 & +2 \\ -3 & -2 & 0 \\ 0 & -3 & -5 \end{pmatrix}, E_2 - BA = \begin{pmatrix} -6 & -1 \\ 5 & -3 \end{pmatrix}. \quad (25)$$

$E_3 - AB$, $E_2 - BA$ are invertible.

$$\left|\lambda E_3 - AB\right| = \lambda^{3-2} \left|\lambda E_2 - BA\right|. \quad (26)$$

3.2.4. When Row Number, Column Number of two Matrices are Completely Different. Let $A \in F^{m \times n}, B \in F^{s \times t}$, where any two of m, n, s, t is unequal , suppose $m \le s$, $n \le t$, $\begin{pmatrix} A & \\ & C \end{pmatrix}$ and B have the same size.

In particular, when $R\begin{pmatrix} A & \\ & C \end{pmatrix} = R(B)$, matrix A can be seen as a sub block of matrix B.

n>t, when mn $\neq$ st, $\begin{pmatrix} A \\ A_1 \end{pmatrix}$ and (B, B_1) have the same size. Here it is no longer to be discussed.

3.3. Relation between Matrix using Matrix Decomposition

Matrix decomposition is an effective tool to realize the large-scale data processing and analysis. As we all know, proposition conditions more special, the conclusion is more perfect, but this conclusion is not commonly used, the following statement mainly focuses on general matrix decomposition structure, in order to get the general, widely used conclusions.

3.3.1. Addition Decomposition. If matrix A can be decomposed into two or several matrix such as A is sum of some matrix and unit matrix , then we can use the matrix decomposition formula to solve a series of related problems such as matrix and A can be commutative; power of A ; or eigenvalue of A.

Each square matrix of order n is usually uniquely presented by the sum of the symmetric matrix and skew-symmetric matrix. That is for every matrix A=S+T, where S is a symmetric matrix and T is a skew-symmetric matrix.

The decomposition gives theory basis and method by using the general matrix to construct symmetric matrices, skew-symmetric matrices.

$A = \frac{A + A^T}{2} + \frac{A - A^T}{2}$, where $\frac{A + A^T}{2}$ is a symmetric matrix and $\frac{A - A^T}{2}$ is a skew-symmetric matrix.

Each square matrix of order n is usually represented by the sum of the scalar matrix and matrix of trace 0.

Each square matrix of order n in complex field is usually represented by the sum of the nilpotent matrix B and an diagonal matrix C and BC=CB.

Each matrix of rank r is usually represented by the sum of the r-th matrices of rank 1.

Each square matrix of order n is usually represented by the linear sum of

$$E_{ij} = e_i e_j^T, where\ e_j = \left(0, \dots, 0, \overset{(j)}{1}, 0, \dots, 0\right)^T, i, j = 1, 2, \dots, n. \quad (27)$$

Such as $A = (a_{ij}) \in F^{n\times n}$, then $A = \sum_{1\le i, j\le n} a_{ij} E_{ij}$.

This decomposition is based on the linear space theory and shows the importance of the matrix of shaped like E_{ij} .

3.3.2. Product Decomposition. If a matrix A can be decomposed into the product of two matrices (such as matrix A of rank 1) , or similarity diagonalization (such as A has n linearly independent eigenvectors) , then the power of A, the polynomial of A and the rank of A can be calculated by using the product decomposition of A.

Case 1: Some general product decomposition, such as:

Any matrix of rank 1 can be factorized into the product of two vectors of rank 1.

Example 5: Let $A = \begin{pmatrix} -1 & 2 & 3 \\ 1 & -2 & -3 \\ 2 & -4 & -6 \end{pmatrix}$ to calculate A^{2014} .

Because R(A)=1 , $A = \alpha\beta^T = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} -1 & 2 & 3 \end{pmatrix}$, where $\alpha = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \beta = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$

$$A^2 = (\alpha\beta^T)(\alpha\beta^T) = \alpha(\beta^T\alpha)\beta^T = 9A \quad (28)$$

$$A^3 = A^2 A = 9^2 A, \quad (29)$$

$$A^{2014} = 9^{2013} A = \begin{pmatrix} -9^{2013} & 2\cdot 9^{2013} & 3\cdot 9^{2013} \\ 9^{2013} & -2\cdot 9^{2013} & -3\cdot 9^{2013} \\ 2\cdot 9^{2013} & -4\cdot 9^{2013} & -6\cdot 9^{2013} \end{pmatrix}. \quad (30)$$

Each square matrix A is usually represented by the product of matrices such as

$$E_n + a_{ij}E_{ij}\ \text{, where } E_{ij} = e_i e_j^T, e_j = \left(0, \dots, 0, \overset{(j)}{1}, 0, \dots, 0\right)^T, i, j = 1, 2, \dots, n\ . \quad (31)$$

Case 2: Some famous special product decomposition, such as triangle decomposition, schur decomposition, fitting decomposition, singular value decomposition, non-negative matrix factorization and so on.

Triangle Decomposition

LU Decomposition: Each square matrix A is usually represented by the product of lower triangular L and an upper triangular U, such that A=LU.

In the calculation of determinant, we often reduced Complex determinant to Upper (lower) triangular determinant using the determinant properties or by row (column) expansion theorem. LU decomposition, each matrix is closely related to upper (lower) triangular matrix.

QR Decomposition: For $A \in C^{n\times n}$, then there exist orthonormal matrix Q, upper triangular matrix R, such that $A = QR$.

Example 6: Let $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 2 & 2 \\ 0 & 1 & 3 \end{pmatrix}$, to calculate QR decomposition of A.

Let $a_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, a_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, a_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, use the unit orthogonal method of Schmidt , then

$$b_1 = a_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, q_1 = \frac{b_1}{|b_1|} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} a_1 \quad (32)$$

$$b_2 = a_2 - \frac{(b_1, a_2)}{(b_1, b_1)} b_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, q_2 = \frac{b_2}{|b_2|} = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = \frac{1}{3} a_2 \quad (33)$$

$$b_3 = a_3 - \frac{(b_1, a_3)}{(b_1, b_1)} b_1 - \frac{(b_2, a_3)}{(b_2, b_2)} b_2 = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 2 \end{pmatrix}, q_3 = \frac{b_3}{|b_3|} = \begin{pmatrix} -\frac{\sqrt{2}}{6} \\ -\frac{\sqrt{2}}{6} \\ \frac{2\sqrt{2}}{3} \end{pmatrix} = \frac{\sqrt{2}}{6} a_1 - \frac{\sqrt{2}}{3} a_2 + \frac{\sqrt{2}}{3} a_3 \quad (34)$$

Make

$$Q = (q_1, q_2, q_3) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{2}{3} & -\frac{\sqrt{2}}{6} \\ -\frac{1}{\sqrt{2}} & \frac{2}{3} & -\frac{\sqrt{2}}{6} \\ 0 & \frac{1}{3} & \frac{2\sqrt{2}}{3} \end{pmatrix}, C = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{\sqrt{2}}{6} \\ 0 & \frac{1}{3} & -\frac{\sqrt{2}}{3} \\ 0 & 0 & \frac{\sqrt{2}}{3} \end{pmatrix} \quad (35)$$

$$R = C^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{\sqrt{2}}{6} \\ 0 & \frac{1}{3} & -\frac{\sqrt{2}}{3} \\ 0 & 0 & \frac{\sqrt{2}}{3} \end{pmatrix}^{-1} = \begin{pmatrix} \sqrt{2} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 3 & 3 \\ 0 & 0 & \frac{3}{\sqrt{2}} \end{pmatrix} \quad (36)$$

Therefore QR decomposition of A is

$$A = QR = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{2}{3} & -\frac{\sqrt{2}}{6} \\ -\frac{1}{\sqrt{2}} & \frac{2}{3} & -\frac{\sqrt{2}}{6} \\ 0 & \frac{1}{3} & \frac{2\sqrt{2}}{3} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 3 & 3 \\ 0 & 0 & \frac{3}{\sqrt{2}} \end{pmatrix}. \quad (37)$$

Schur Decomposition:

let $A \in C^{n \times n}$, then there exist unitary matrix U , such that $A = UTU^H$, where T is an upper triangular matrix which diagonal elements are all eigenvalue of A .

Fitting Decomposition:

Let $A \in F^{n \times n}$, then there exists an invertble matrix T, such that $A = T diag\{D, N\} T^{-1}$, where D is an invertible matrix, N is a nilpotent matrix.

Square matrix A of order n itself may not be invertble or nilpotent, but A is similar to a Block diagonal matrix consisting of invertble blocks and nilpotent blocks.

Singular Calue Decomposition:

Let $A \in C^{m \times n}$, then A is usually represented by the product of unitary matrix U of order m, semi-positive definite m×n diagonal matrix S and unitary matrix V of order n , such that $A = USV^*$.

Example 7: Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$, to calculate Singular value decomposition of A.

Because $A^T A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix}$, then the eigenvalue of $A^T A$ are $\lambda_1 = 3, \lambda_2 = 1, \lambda_3 = 0$.

The corresponding eigenvectors are $\xi_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \xi_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \xi_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$,

$$V = \left(\frac{\xi_1}{|\xi_1|}, \frac{\xi_2}{|\xi_2|}, \frac{\xi_3}{|\xi_3|} \right) = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \end{pmatrix} \tag{38}$$

Then

$$V^T \left(A^T A\right) V = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{pmatrix} \text{, where } \Sigma = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{pmatrix}. \tag{39}$$

$$U = AV_1 \Sigma^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \tag{40}$$

So the Singular value decomposition of A is

$$A = U\,(\Sigma, O)\,V^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix}. \tag{41}$$

Non-negative Matrix Factorization:

For an arbitrarily given non negative matrix A, there are a non-negative matrix U and a non-negative matrix V, such that A=UV.

That is to say that A non-negative matrix can be factorized into the product of two non-negative matrices.

Non-negative matrix factorization algorithm provides a new idea for matrix decomposition. Non-negative matrix factorization algorithm in the analysis and processing of images , text clustering and data mining , speech processing, robot control , biomedical engineering and chemical engineering has important application , in addition , Non-negative matrix factorization algorithm in the data signal analysis and complex object recognition have a very good application.

4. Conclusion

Firstly, some property of the transpose matrix or adjoint matrix A^* of A may depend on the original matrix A. Such as the rank, determinant, especially adjoint matrix A^* of A is linear combination of power of A.

Secondly , this paper simplified formula about three kinds of equivalence relations between matrices of the same type , such as A , B be matrices of the same type , A=PBQ, where P , Q are invertible matrices .

Thirdly, it discussed the relations between matrices of different types with the operation of the matrix. And using addition and multiplication operation of matrices to establish relationship between general matrix and symmetric matrix (skew-symmetric matrix), upper (lower) triangular matrix, invertible matrix, nilpotent matrix.

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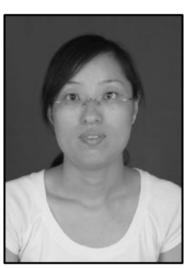

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