Inverse Problem of LQ-PID Control for TCP/AQM Routers

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Abstract

In this paper, we take into considerations of the inverse problem for LQ-PID control. Solving the inverse problem, explicit formulas are developed such that the closed loop system is optimal by an LQ-PID controller. The inverse problem of LQ-PID control is analytically formulated via the frequency-domain and algebraic characterizations of LQR with respect to special forms of the state weighting matrix Q. *We apply the LQ-PID control to AQM(Active Queue Management) routers from the viewpoint of the inverse problem. In the controller design procedure, the practical effectiveness of the proposed analysis is demonstrated by the numerical simulation of AQM routers.*

Keywords: *LQ-PID Control, Inverse Optimal Problem, AQM Router*

1. Introduction

In optimal control theory, the interest for the IP (inverse problem: When is a given control law *G* optimal for some state weighting *Q* of a performance index?) has been actively grown since Kalman's outstanding paper [1] was published, dealing with the properties of LQR (Linear Quadratic Regulator) [2]. Thereafter the researches have been extended to linear H_2 [3], H_∞ [4] control.

We deal with the IP for LQ-PID (Linear Quadratic-Proportional Integral Derivative) control, in which the output feedback PID control is successively linked to LQR with the full state feedback augmenting a new state for second order systems [5, 6]. Recent two papers have shown the attempts to relate the performance indices to the design specifications in time and frequency domains for LQ-PID control [5, 6].

In the LQ-PID control, the purpose of this research is not only to present the allowable region of the feedback control gain such that a closed loop system is optimal, but to find the formula of state weighting matrix *Q* to give the same optimal control gain. By the analysis of the IP, we, hence, clarify the relation between the optimal feedback control gain and the corresponding performance index. The allowable region indicates obviously the condition of the feedback control gain to make the controlled system optimal in the sense of the LQ-PID control. From the relation, one can then conversely check if the optimal controller makes the closed loop poles place in the desired region or satisfy the desired specifications. In this paper, we set the problem under the simple and familiar restrictions without the loss of the generality, as symmetric and a p.s.d (positive semi-definite) *Q* and a positive scalar ρ. Whereas the symmetric and p.s.d *Q* can be variously factorized, we take account into the special forms to be partitioned by diagonal and single row matrices. The case of the diagonal partitioned matrix was dealt with for LQ-PID control since this is relatively more convenient to directly match the performance index to the time domain specifications such as overshoot, rising time, and settling time [5]. On the other hand, it is very useful to associate the frequency loop shaping method with the single row partitioned matrix for frequency domain specifications of LQ-PID control [1, 6]. That is why among the general p.s.d *Q*s, we consider the two special cases in this paper. From the results, it is expected that the result of this research is helpful to determine the allowable set of the performance index in order to meet the time and frequency domain specifications in practice. To formulate the IP, we utilized the characterizations of optimality, so-called as algebraic and frequency-domain characterizations for the analytic clarification and the simplicity of the formula in such cases, rather than the numerical approaches by LMI (Linear Matrix Inequality) [1, 7].

We apply the LQ-PID control to AQM routers supporting TCP flows from the viewpoint of the IP, on which more attention has recently focused as important issues for congestion controls of the information and sensor networks [8, 9].

The paper is organized as follows. We introduce the LQ-PID formulation by transforming the PID control into LQ approach for the comprehension of readers in Section 2. The IP of the LQ-PID control is analyzed by algebraic and frequency characterizations with respect to such cases of the weighting matrix *Q* in the next section. The effectiveness of the proposed analysis is shown practically by the numerical simulation of AQM routers in Section 4. Finally, some concluding remarks are contained.

2. LQ-PID Control

Briefly, we introduce the LQ-PID formulation in the paper of Suh and Yang [5], in which the optimal feedback control law was successfully related to the PID control for the second order system by augmenting the integral of the output variable as a new state variable.

Consider the following second order model:

$$\frac{d^2 y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = c\omega_n^2 u(t). \tag{1}$$

where $y(t)$, $u(t)$, ζ and ω_n are the output variable, the control variable, the damping ratio, and the natural frequency, respectively. The augmented plant is represented as Eq. (2).

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega_n^2 & -\zeta\omega_n^2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ c\omega_n^2 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}. \tag{2}$$

The quadratic performance criterion is the same as the convenient form in optimal control as following

$$J = \frac{1}{2}\int_0^\infty \left(x(t)^T Q x(t) + \rho u(t)^T u(t) \right) dt. \tag{3}$$

, assuming that the weighting factor Q is symmetric and p.s.d, and ρ is positive scalar. The linear feedback control law is obtained as follows

$$u(t) = -Gx(t), \tag{4}$$

where G is $\rho^{-1}B^T K$ and $K=K^T$ is the solution of the ARE under the condition that (A,Q) be observable. The design parameter matrix G is then as following

$$G = \begin{bmatrix} g_0 & g_1 & g_2 \end{bmatrix} = \frac{c\omega_n^2}{\rho} \begin{bmatrix} K_{20} & K_{21} & K_{22} \end{bmatrix}. \tag{5}$$

The optimal control law of Eq. (6) is represented as the following PID control formula:

$$u(t) = -\frac{c\omega_n^2}{\rho}\left[K_{20}\int_0^t y(\tau)\,d\tau + K_{21}y(t) + K_{22}\frac{dy(t)}{dt}\right]. \tag{6}$$

The tuning parameters of the LQ-PID controller are hence Q and ρ, obtained by solution K.

3. Inverse Problem of LQ-PID Control

In this paper, we deal with the inverse optimal problem of LQ-PID control, by which the p.s.d and symmetric Q is determined, given G and ρ to satisfy the performance specifications. Since the analysis of the IP depends on the type of Q, we regard following two cases of factorization of $Q=N^TN$ without the loss of generality among many kinds of partitioned matrices of Q.

Case 1 is based on the diagonal matrix Q factorized by a diagonal N, which was applied to Suh and Yang's paper since it has the advantage to give the simple intuition with respect to relation between the quadratic performance and the state weighting Q. The other case of Q is the single row partitioned matrix, which makes it easy to analyze the optimal feedback system in frequency domain, approached by Yang and Suh [6]. Hence, the cases are presented as follows

Case 1) $Q = \begin{bmatrix} n_0 & 0 & 0 \\ 0 & n_1 & 0 \\ 0 & 0 & n_2 \end{bmatrix}^T \begin{bmatrix} n_0 & 0 & 0 \\ 0 & n_1 & 0 \\ 0 & 0 & n_2 \end{bmatrix}$

Case 2) $Q = [n_0 \quad n_1 \quad n_2]^T [n_0 \quad n_1 \quad n_2]$

This inverse optimal control problem is characterized by two approaches as analyzed in Kalman's paper [1]. One is based on ARE, called as algebraic characterization of optimality. This nonlinear ARE and Eq. (5) can be transformed into the LME (Linear Matrix Equality) in term of the solution K as follows

$$KA + A^TK + Q - \rho G^TG = 0. \tag{7}$$

Substituting Eq. (5) to Eq. (7), the elements of the diagonal Q are obtained as

$$n_0^2 = \rho g_0^2\ ,\ n_1^2 = \rho g_1^2 + 2b\frac{\rho}{c}g_1 - 2\rho\, g_0 g_2 - 2a\frac{\rho}{c}g_0\ ,and\quad n_2^2 = \rho g_2^2 + 2a\frac{\rho}{c}g_2 - 2\frac{\rho}{c}g_1. \tag{8}$$

For the Case 2, we use the frequency-domain characterization of optimality as follows

i) G be a stable control law

ii) $\rho\big(1 + g_{LQ}(-j\omega)\big)\big(1 + g_{LQ}(j\omega)\big) = \rho + \big(N(-j\omega - A)^{-1}B\big)^T N\big((j\omega - A)^{-1}B\big)$

where
$$g_{LQ}(s) = G(sI - A)^{-1}B = \frac{(g_2 s^2 + g_1 s + g_0)c\omega_n^2}{s(s^2 + 2\zeta\,\omega_n s + \omega_n^2)}. \tag{9}$$

i) and ii) are the conditions for the stability and FDE (Frequency Domain Equivalent) of ARE, respectively.

According to Theorem 9 in Kalman's paper [1], necessary and sufficient conditions for optimality of G are consisted of the following stability condition I of the closed loop system and the existence condition II of p.s.d matrix Q, which are modified from the above two conditions i) and ii).

I) $\psi_k(s)$ satisfies the Routh-Hurwitz conditions.

II) $\left|\psi_k(j\omega)\right|^2 - \left|\psi(j\omega)\right|^2 = \rho^{-1}\left|\Psi(j\omega)\right|^2$

where $\psi_k(s)$, $\psi(s)$, and $\Psi(s)$ denote the characteristic equations of the closed and open loop system, and the numerator polynomial of $N(sI\text{-}A)^{-1}B$, respectively.

In Case 2, the simple relation between Q and G is clarified by Eq. (10), derived by substituting Eq. (9) to the condition II).

$$\begin{cases} \left(4\zeta\,\omega_n g_2 + c\omega_n^2 g_2^2 - 2g_1\right)(j\omega)^4 + \left(4\zeta\,\omega_n g_0 + 2c\omega_n^2 g_2 g_0 - 2\omega_n^2 g_1 - c\omega_n^2 g_1^2\right)(j\omega)^2 + c\omega_n^2 g_0^2 \\ = \dfrac{c\omega_n^2}{\rho}\left[n_2^2(j\omega)^4 + (2n_2n_0 - n_1^2)(j\omega)^2 + n_0^2\right] \end{cases} . \tag{10}$$

For the stability condition I, $\psi_k(s)$ should satisfy the Routh-Hurwitz. If the condition II is satisfied by Eq. (11), then a real number N is available since it has no imaginary roots in Eq. (10).

$$\begin{cases} \left(c\omega_n^2 g_0\right)^2 \geq 0 \\ \left(4\zeta\,\omega_n g_2 + c\omega_n^2 g_2^2 - 2g_1\right)c\omega_n^2 \geq 0 \\ Either\ \left(2\omega_n^2 g_1 + c\omega_n^2 g_1^2 - 4\zeta\,\omega_n g_0 - 2c\omega_n^2 g_2 g_0\right)c\omega_n^2 \geq 0 \\ or\ \left(2\omega_n^2 g_1 + c\omega_n^2 g_1^2 - 4\zeta\,\omega_n g_0 - 2c\omega_n^2 g_2 g_0\right)c\omega_n^2 \\ \geq -2\rho\sqrt{\left(c\omega_n^2 g_0\right)^2\left(4\zeta\,\omega_n g_2 + c\omega_n^2 g_2^2 - 2g_1\right)c\omega_n^2} \end{cases} . \tag{11}$$

Suppose that we take the classical second order system, where c, ζ and ω_n are all positive real numbers. The stability conditions can be summarized as $g_0>0$, $g_2>-2\zeta/(c\omega_n)$ and $g_1>g_0/((2\zeta\omega_n+c\omega_n^2 g_2))-1/c$. For the existence of the solution of the IP, they are parabolically related between g_1 and g_2, combined with the stability condition for g_2.

Finally, the formulas of the state weighting Q are described Eq. (8) and (10) with respect to Case 1 and 2, respectively. And the conditions are that the right side of Eq. (8) is positive and inequality condition of Eq. (11) is satisfied in order to exist the optimal solution of LQ-PID control

4. A Case Study for TCP/AQM

We implement the LQ-PID control to TCP/AQM systems in the simulations to verify the practical effectiveness of the proposed analysis. The same example was dealt with in Hollot et al's papers [10]. In the simulation, we consider 60 of TCP flows $\mathbf{N_{aqm}}$, 3750 packets/secs link capacity $\mathbf{C_{aqm}}$, and the propagation delays $\mathbf{T_p}$ for the flows range uniformly between 160 and 240msec. The maximum buffer $\mathbf{q_{max}}$ and window size $\mathbf{W_{max}}$ are 800 and 20 packets. For linearization, operating points, $\mathbf{W_0}$, $\mathbf{q_0}$, $\mathbf{p_0}$, and $\mathbf{R_0}$ are 15 packets, 175 packets, 0.0084 and 0.247sec, respectively.

$$P(s) = \frac{\dfrac{{C_{aqm}}^2}{2N_{aqm}}}{\left(s + \dfrac{2N_{aqm}}{R_0^2 C_{aqm}}\right)\left(s + \dfrac{1}{R_0}\right)} e^{-R_0 S} = \frac{1.172 * 10^5}{s^2 + 4.58s + 2.132} e^{-0.247s} .$$

Considering the IP for Case 1 and 2, we obtained the p.s.d Q and Q_{diag}, given the LQ-PID control gain $G=10^{-4*}[0.368\ 0.791\ 0.173]$ to yield the proper loop shape in the paper [11] as following

$$Q_{diag} = 10^{-6}\begin{bmatrix} 0.136 & 0 & 0 \\ 0 & 0.499 & 0 \\ 0 & 0 & 0.030 \end{bmatrix}, Q = 10^{-3} \times \begin{bmatrix} 0.368 \\ 0.791 \\ 0.173 \end{bmatrix} \begin{bmatrix} 0.368 & 0.791 & 0.173 \end{bmatrix} \times 10^{-3}.$$

$G_{Huang}=10^{4*}[3.086\ 6.630\ 1.447]$ is selected by Huang's method for a comparison of the optimality [19]. We realized that the control law G_{Huang} isn't optimal in the viewpoint of LQ-PID approach since the solution isn't obtainable in the IP, analysed in this paper.

5. Conclusion

In this paper, we have analyzed the IP for LQ-PID control. The IP is to determine the weighting Q corresponding to the given G and R for the optimality. Among many factorizations of a p.s.d and symmetric Q, we handle the particular two cases such as the diagonal (Case 1) the single row partitioned (Case 2) matrices of Q. In such cases, we analytically formulated the simple relation to connect G with Q by ARE and FDE, based on Kalman's paper. From the formulas, the allowable conditions of the LQ-PID control gain were obtained to make the controlled feedback system optimal.

We applied the LQ-PID control to the AQM routers from the viewpoint of the IP. The numerical case study for the TCP/AQM router has shown the various manners to determine practically the tuning parameter Q with the various factorizations by the analysis of the IP, given the proposed LQ-PID control gain G.

Consequently, it is expected that this paper provides the effective way to determine the weighting Q by the analysis with respect to the relationships between the design parameters and the weighting factors via the IP in order to meet design specification in time and frequency domains.

References

[1] R. E. Kalman, J. Bas. Eng. 86 **(1964)**.

[2] A. Jameson and E. Kreindler, SIAM J. Control, 11 **(1973)**.

[3] T. Fujii, IEEE Trans. Automat. Control, 32, 11 **(1987)**.

[4] T. Fujii and P. P. Khargonekar (Eds.), IEEE. Proceedings of the 27th IEEE Conference on Decision & Control, **(1988)** December 7-9; Austin (Texas), U.S.A.

[5] B. S. Suh and J. H. Yang, J. Chem. Eng. Japan, 38, 5 **(2005)**.

[6] J. H. Yang and B. S. Suh, J. Chem. Eng. Japan, 40, 7 **(2007)**.

[7] S. Boyd, L. E. Ghaoui, E. Feron and V. Balakrishnan, in Linear Matrix Inequalities in System and Control Theory, SIAM, Philadelphia **(1994)**.

[8] J. Y. Heo, J. The Inst. of Webcasting, Internet and Telecommunication. Korea, 11, 6, **(2011)**.

[9] S. C. Kim, J. The Inst. of Webcasting, Internet and Telecommunication. Korea, 9, 2, **(2009)**.

[10] C. V. Hollot, V. Misra, D. Towsley, and W. Gong, IEEE Trans. Automat. Control, 47, 6 **(2002)**.

[11] H. P. Huang, M. W. Lee, and C. L. Chen, J. Chin. Inst. Chem. Eng. 31 **(2000)**.

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