International Journal of Multimedia and Ubiquitous Engineering
Vol. 4, No. 2, April, 2009

An improved g-centroid location algorithm for Ptolemaic graphs

Prakash Veeraraghavan
Department of Computer Science and Computer Engineering
La Trobe University, 3086, Victoria, Australia
P.Veera@latrobe.edu.au

Abstract

We have presented an $O(m^2)$ time algorithm for locating the g-centroid for Ptolemaic graphs, where n is the number of edges and m is the number of vertices of the graph under consideration [6]. If the graph is sparse (i.e. $m = O(n)$) then the algorithm presented will output the g-centroid in quadratic time. However, for several practical applications, the graph under consideration will be dense (i.e. $m \approx O(n^2)$) and the algorithm presented will output g-centroid in $O(n^4)$ time. In this paper, we present an efficient $O(n^3)$ time algorithm to locate the g-centroid for dense Ptolemaic graphs.

Keywords: *g-centroid, g-convexity, ptolemaic graph*

1. Introduction

In this paper, we present an improved version of our earlier algorithm given in [6] to locate the g-centroid for ptolemaic graph.

The concept of *g-convexity* in graphs was introduced and studied under different context, mostly based on similar concepts in Eucledean space and topology. Though different type of convexity exists in the literature, the most important and natural type of convexity is the *geodetic-convexity* (*g-convexity* for short). In [5] this was studied under the name *intervel-convexity*. It was also demonstrated that g-convexity can be applied to distributed computing and generalized hypercube architecture for parallel processing. Besides this, g-convexity and g-centroid (defined through g-convexity) found application in telephone switching center, facility location, distributed computing, information retrieval and power optimization in mobile ad hoc networks [6], [7]. Recently, an application of g-centroid, especially in power optimization for mobile ad hoc networks was presented in [8].

The problem of locating the g-centroid of a graph, defined through g-convexity gained importance due to above practical applications. In [6], Pandu Rangan, Parthasarathy and Prakash have proved that locating the g-centroid for an arbitrary non-trivial disconnected graph is NP-hard, by reducing the problem of finding the maximum clique size of a graph to the problem of locating the g-centroid. In a later paper [8], NP-hardness was established for connected graphs.

We now give the definition of g-centroid through the g-convexity for graphs.

Standard graph-theoretic terms not defined here can be found in Bondy and Murty [1], Buckley and Harary [2].

Definition 1. 1 *A set* $S \subseteq V$ *is* **geodetic convex** *(***g-convex** *for short) if for every pair of vertices* $u, v \in S$, $I(u,v) \subseteq S$, *where* $I(u,v)$ *= {* $x : x$ *lies in any* $u - v$ *geodesic path}*

A convex set is basically a set of vertices which is "closed" with respect to the flow of information (either routing, control or data packets). Thus for several practical applications, a convex set is more prefered.

From the above definition, it easily follows that a singleton set, vertex pair of an edge, and the whole vertex set $V(G)$ are g-convex sets of G. We call them as trivial g-convex sets. Also if S is a clique (S induces a complete sub graph of G), then S is a g-convex set of G.

Definition 2. 2 *Let* $G = (V, E)$ *be any connected graph. For* $v \in V$, *the* **g-weight** $w(v)$ = *max {*$|S|$*:* S *is a g-convex set of* G *not containing* v*}. Let* $gc(G)$ = *min{* $w(v)$ *:* $v \in V$ *} (where* $|S|$ *denotes the number of elements or the cardinality of the set* S*). Then* $gc(G)$ *is called the* **g-centroidal number** *of* G *and the vertices* v *for which* $w(v) = gc(G)$ *are called the* **g-centroidal** *vertices. The* **g-centroid** $C_g(G)$ *is the set of all g-centroidal vertices of* G *(i.e. g-centroid is a set of vertices which satisfies the min-max relation).*

For $v \in V(G)$, we denote by $S_v = S_v(G)$, any maximum g-convex set of G not containing v.

we now give the definition of the "neighbourhood structure" of a graph and some special classes of graphs considered in this paper.

The **i-th neighbourhood** of a vertex u is defined as $N_i(u) = \{x : d(u,x) = i\}$ and we denote $N_1(u)$ by simply $N(u)$.

Definition 3.3 *Let* $u \in V(G)$ *and* $y \in N_i(u)$, *for some* $i \geq 1$. *Then* y *is a* **successor** *(or a* **child***) of* x *with respect to* u *if* $x \in N_{i-1}(u)$ *and* $xy \in E$. x *is also called a* **parent** *or a* **predecessor** *of* y.

Now consider the successor relation with respect to u on G. Let $y \in N_j(u)$. We say y is a **descendant** of $x \in N_i(u)$ if $i < j$ and there exists an $x - y$ geodesic path of length $j - i$. In this case x is also called an **ancestor** of y.

By a neighbourhood structure of G with respect to u, we mean the predecessor- successor relation on $V(G)$ defined with respect to u.

We now give the definitions of some special classes of perfect graphs considered in this paper.

A graph is **chordal** (also called **triangulated**) if every cycle of length strictly greater than three possesses a chord, that is an edge joining two non consecutive vertices of the cycle.

A **ptolemaic graph** is a connected graph such that for every four vertices v_1, v_2, v_3, v_4, the following inequality is satisfied:

$$d(v_1,v_2) * d(v_3,v_4) \leq d(v_1,v_3) * d(v_2,v_4) + d(v_1,v_4) * d(v_2,v_3)$$

A graph is a **distance hereditary graph** if every two vertices have the same distance in every connected induced sub graph containing both.

2. Preliminary results

In this section we give some preliminary results used to prove the main theorem.
The following results are due Howorka.

Proposition 1. (Howorka [4]) 4 *A graph is ptolemaic if and only if it is distance hereditary and chordal*

Proposition 2. (Howorka [3]) 5 *Given a graph $G = (V, E)$, the following three statements are equivalent.*

- G is distance hereditary
- Every cycle in G with five or more vertices has two intersecting chords
- Every induced path (chordless path) in G is a geodesic path

We use the second equivalent statement for proving most of our results in this paper.
Let $u \in V(G)$. Consider the neighbourhood structure of G with respect to u. Let $x \in N_i(u)$, $i \geq 1$. We define $A(x)$ to be the set of all ancestors of x in $N(u)$ with respect to u. If $x \in N(u)$, then we define $A(x) = \{x\}$.
The following results are due to Pandu Rangan, Parthasarathy and Prakash [6].

Proposition 3. 6 *Let G be a chordal graph and $u \in V$. Let $x, y \in N_i(u)$. If x and y are adjacent, then either $A(x) \subseteq A(y)$ or $A(y) \subseteq A(x)$ or $A(x) = A(y)$*

Proposition 4. 7 *Let G be a ptolemaic graph. Let M be any maximal clique of $< N(u) >$. For any x, if $A(x) \cap M \neq \emptyset$, then $A(x) \subseteq M$. That is $A(x)$ is completely contained in a maximal clique of $< N(u) >$*

Based on the above two propositions, for any vertex u of G, the maximal g-convex set not containing u can be found as follows:

Proposition 5. 8 *Let G be a ptolemaic graph and $u \in V(G)$. Let M be a maximal clique of $< N(u) >$. Let $D(M) = M \cup \{ y : y$ is a descendant of some $x \in M \}$. Then $D(M)$ is a g-convex set of G and $S_u = D(M)$ for some maximal clique M of $< N(u) >$*

The above property was exploited in [6] to locate the g-centroid for ptolemaic graphs in $O(m^2)$ time. A brief outline of their algorithm is presented below for completeness:

Ptolemaic graphs form a proper sub class of chordal graphs. Thus for each vertex u, $< N(u) >$ is chordal. It is well-known that any chordal graph on n vertices has exactly $O(n)$ maximal cliques. Therefore $< N(u) >$ has at most $O(d(u))$ maximal cliques, where $d(u)$ is the degree of the vertex u. For each maximal clique, finding $D(M)$ will take $O(n+m)$ time. Thus finding the weight of u takes $O(d(u)(n+m))$ time and hence the g-centroid can be outputted in $O(m^2)$ time.

If $m = O(n)$, then the above straight forward algorithm takes only $O(n^2)$ time to output the g-centroid. However in many practical applications, the graph under consideration will be dense, that is, the number of edges may not be of linear order on the number of nodes. In this paper, we present an efficient algorithm which will take only $O(n^3)$ time to locate the g-centroid for ptolemaic graphs. We follow the same technique presented in [6]. However, instead of finding $D(M)$ for each maximal clique M of $< N(u) >$ for a vertex u and then finding the $D(M)$ having maximum cardinality, we simply locate a maximal clique M of $< N(u) >$ for which $D(M)$ has the maximum cardinality. This is by defining an auxiliary graph $H(u)$ for every vertex u of G.

Let $G = (V, E)$ be any connected graph and $u \in V$. We construct the auxiliary graph $H(u)$ as follows:

- $V(H(u)) = V(G) - \{u\}$
- Two vertices x and y are adjacent in $H(u)$ if and only if $u \notin I(x, y)$ = { z : z lies in an $x - y$ geodesic path}.

3. Main Results

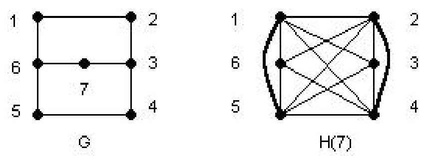
In this section we present our main results which leads to an $O(n^3)$ time algorithm for locating the g-centroid for ptolemaic graphs.

The proof of the following proposition is immediate from the definition of $H(u)$.

Proposition 6. 9 *For each vertex u of G,*

If $xy \in E(G)$, and x and y are different from u, then $xy \in E(H(u))$.

1. $H(u)$ has an induced subgraph isomorphic to $< N(u) >$.
2. $H(u)$ is complete if and only if u is a simplicial vertex (that is $< N(u) >$ is complete) of G.
3. If S is a g-convex set of G not containing u, then S induces a complete subgraph in $H(u)$



In H(7), {1,2,3,4} is a clique, but it is not a convex set in G

The converse of (4) in the above proposition is not true in general. This fact is illustrated in the above figure. If the converse also holds for certain classes of graphs, then $w(u)$ = $\omega(H(u))$, the maximum clique size of $H(u)$. Thus if the maximum clique problem is polynomial for all $H(u)$'s then $C_g(G)$ can be outputted in polynomial time.

We now show that the converse holds for ptolemaic graphs. Before proving this, we give a lemma.

Lemma 1. 10 *Let G be a ptolemaic graph and $u \in V(G)$. Then $x, y \in H(u)$ are adjacent in $H(u)$ if and only if there exists a maximal clique M of $< N(u) >$ such that $x, y \in D(M)$.*

Proof Let x and y be adjacent in $H(u)$. We prove that $x, y \in D(M)$ for some M .This is by considering various possibilities for x and y.

Case 1: Both x and y belong to $N_G(u)$ (that is x and y are adjacent to u in G.)

If x and y are not adjacent in G, then x, u, y is an $x-y$ geodesic path through u. Thus $u \in I(x, y)$. This implies that x, y are not adjacent in $H(u)$, a contradiction. If M is a maximal clique of $< N(u) >$ containing both x and y, then $x, y \in D(M)$ (at least one M will exist, otherwise { x, y } itself is a maximal clique of $< N(u) >$ containing both).

Case 2: $x \in N(u)$ and $y \notin N(u)$

If $x \in A(y)$, then if M is any maximal clique of $< N(u) >$ containing x (it may be $A(y)$), then by Proposition 4, $A(y) \subseteq M$. Thus $x, y \in D(M)$.

We now discuss the case when $x \notin A(y)$. Here we show that $< x \cup A(y) >$ is complete.

Since $x \in N(u)$, $d(x, y) \geq d(y, u) - 1$. If equality holds, then by the definition of descendants, y is a descendant of x and hence $x \in A(y)$, which contradicts our assumption that $x \notin A(y)$. Therefore $d(x, y) \geq d(y, u)$.

If $d(x, y) > d(y, u)$, then if P is any $y-u$ geodesic, then $P \cup \{x\}$ is an $x-y$ geodesic path of length $d(y, u)$ + 1. Thus $u \in I(x, y)$ and consequently x and y are non adjacent in $H(u)$. Therefore we have $d(x, y) = d(y, u)$.

Our next claim is to prove that x is adjacent to every vertices of $A(y)$. To establish this claim, we first show that x is adjacent to at least one vertex of $A(y)$.

Let P be any $x-y$ geodesic path of length $d(u, y)$. Let z be any arbitrary vertex in $A(y)$ and Q be an $y-z$ geodesic. By deleting the portions common to P and Q, we have a cycle containing u, z, x, y. Since G is chordal, we have $zx \in E$.

We now show that x is adjacent to all the vertices of $A(y)$. If possible, let x be adjacent to z of $A(y)$ but not adjacent to z_1 of $A(y)$.

Let w be a common descendant of z and z_1 in $N_2(u)$. Such a w exists, for if not, let Q_1 and Q_2 be geodesics joining y, z and y, z_1. Let z' precede z in Q_1 and $z_{1'}$ precede z_1 in Q_2. Then we have a cycle $Q_1 \cup Q_2$ containing z, z_1, z', $z_{1'}$. Since G is chordal, either $zz_{1'} \in E$ or $z_1 z' \in E$. That is z and z_1 have a common neighbour in $N_2(u)$.

Now consider the cycle u, x, z, w, z_1, u. This is a 5-cycle with zz_1, uz as chords. Since G is ptolemaic, this cycle must have two intersecting chords. Thus $z_1 x \in E$, a contradiction. This proves that x is adjacent to all the vertices of $A(y)$ and hence $A(y) \cup \{x\}$ is a clique of $< N(u) >$. If M is a maximal clique of $< N(u) >$ containing $A(y) \cup \{x\}$, then x, $y \in D(M)$ (at least one such M will exist, otherwise, $A(y) \cup \{x\}$ itself is the target maximal clique).

Case 3: Both x and $y \notin N(u)$

If $A(x) \cap A(y) = \varnothing$ and there is no edge between a vertex in $A(x)$ and a vertex in $A(y)$, we can easily show that $u \in I(x, y)$, which is a contradiction.

If $z \in A(x)$ and $z_1 \in A(y)$ are adjacent, then as before, we can show that z is adjacent to all the vertices of $A(y)$ and by considering z_1, z_1 is adjacent to all the vertices of $A(x)$. Consequently any $p \in A(x)$ and $q \in A(y)$ are adjacent. That is $A(x) \cup A(y)$ is a clique of $< N(u) >$.

Suppose $A(x) \cap A(y)$ is nonempty. If M is a maximal clique of $< N(u) >$ containing $A(x)$, then by Proposition 4, $A(y)$ is also contained in M. Hence $x, y \in D(M)$

To prove the converse of this proposition, let $x, y \in D(M)$. By Proposition 5, $D(M)$ is a g-convex set of G. From (4) of Proposition 6, $D(M)$ induces a complete subgraph in $H(u)$. Thus $xy \in E(H(u))$

Using this lemma, we can prove the following result:

Proposition 7. 11 *If G is a ptolemaic graph and $u \in V(G)$, then $H(u)$ is chordal.*

Proof: If u is a simplicial vertex of G, then $H(u)$ is complete and hence it is chordal. Let us assume that u is a non-simplicial vertex of G. We prove $H(u)$ is chordal by proving that any cycle of length four or more in $H(u)$ has a chord. Let $C : u_1, u_2, \cdots, u_r, u_1, (r \geq 4)$ be any cycle in $H(u)$

Now $u_1 u_2 \in H(u)$ implies that $u_1, u_2 \in D(M_1)$ for some maximal clique M_1 of $< N(u) >$. Similarly $u_2, u_3 \in D(M_2)$. If $M_1 = M_2$, then $u_1 u_3 \in E(H(u))$ and hence C has a chord. If $M_1 \neq M_2$, then consider u_3, u_4. Let $u_3, u_4 \in D(M_3)$. If $M_2 = M_3$, then as before C has a chord. If M_1, M_2 and M_3 are distinct. Let $m_1 \in A(u_1) \cap (M_1 - M_2)$, $m_2 \in A(u_2) \cap M_1 \cap M_2$, $m_3 \in A(u_3) \cap M_2 \cap M_3$ and $m_4 \in M_3 - M_2$. Then m_1, m_2, m_3, m_4 is a path on four vertices. Thus u, m_1, m_2, m_3, m_4, u is a 5-cycle with um_i, $1 \leq i \leq 4$ as chords. Since G is ptolemaic, this cycle must have two intersecting chords. Thus there is a chord of the form $m_i m_j$ with $i \neq j$, say $m_1 m_3$. Then $u_1 u_3 \in E(H(u))$. Since C is arbitrary, G is a chordal graph.

Proposition 8. *Let G be a ptolemaic graph and $u \in V$. If S is a maximal clique in $H(u)$, then S is a maximal g-convex set of G.*

The proof follows from the structure of G and $H(u)$

Corollary 1. 12 *Let G be a ptolemaic graph. Then for each vertex u of G, $w(u)$ = $\omega(H(u))$, the maximal clique size of $H(u)$*

Using this corollary, we present a polynomial time algorithm to locate the g-centroid for ptolemaic graphs. First we give an algorithm to construct $H(u)$ for each vertex u.

Procedure CONSTRUCT(G)
begin

```
    for each u ∈ V do
        execute BFS(u) to get the successor relation;
        for each x ∈ N(u), I(u,x) = { u , x };
        for each x ∉ N(u), if x_1, x_2, ···, x_r are
            the parents of x, then I(u,x) = x ∪_{i=1}^{r} I(u,x_i);
        for each v do
            if v ∉ I(u,x) then add ux to E(H(v));
        endfor;
    endfor;
return;
```

Complexity analysis for CONSTRUCT

For each vertex u, BFS(u) (Breadth First Search) takes $O(n+m)$ time. In the worst case, $I(u,x)$ can have $O(n)$ vertices. Hence finding $I(u,x)$ takes $O(n)$ time. Therefore finding $I(x,y)$ for every pair x, y takes $O(n^3)$ time. Also $H(u)'s$ can be constructed in $O(n^3)$ time. Thus in total, the procedure CONSTRUCT takes $O(n^3)$ time.

We now give a procedure to locate the g-centroid for ptolemaic graphs.

```
Procedure GCENT(G)
begin
    CONSTRUCT(G);
    for each vertex u ∈ V, w(u) = ω(H(u));
    output the least weighted vertices;
return.
```

Complexity analysis

Since $H(u)$ is chordal, finding $\omega(H(u))$ takes $O(n+m')$ time, where m' is the number of edges in $H(u)$. Clearly $m' \geq m$. But in the worst case, $m' = O(n^2)$. Since there are n vertices in G, finding the weight of all the vertices takes $O(n^3)$ time. Outputting the least weighted vertices takes $O(n)$ time. Therefore outputting the g-centroid takes $O(n^3)$ time.

Theorem 1. 13 *Let G be a ptolemaic graph. Then the g-centroid can be outputted in $O(n^3)$ time.*

4. Conclusion and Future Direction

In [6], Pandu Rangan, Parthasarathy and Prakash have presented a nice and straight forward algorithm for locating the g-centroid for maximal outerplanar graphs, ptolemaic and split graphs. This is by exploiting the structural characterstics of these special classes of perfect graphs. We improve on their earlier algorithm for ptolemaic graphs by defining an auxiliary graph at every vertex. It would be interesting to provide a polynomial time algorithm for other classes of perfect graphs such as interval graphs and cocomparability graphs. Also to classify all the graphs for which the converse of (4) in Proposition 6 is true is an interesting open problem in this direction.

References

[1] J.A.Bondy and U.S.R.Murty, *Graph Theory and Applications*, Macmilon, London (1976).
[2] F.Buckley and F.Harary, *Distances in graphs*, Addison-Wesley, NY (1989).
[3] E.Howorka, A characterization of distance hereditary graphs, *Quart. J. Math.*, Oxford, Vol. 28, (1977) pp. 417-420.
[4] E.Howorka, A characterization of ptolemaic graphs, *J. Graph Theory*, Vol. 5, (1981) pp. 323-331.
[5] H.M.Mulder, The Interval Function of a Graph, *Mathematish Centrum*, Tract. 132, Amsterdam (1981).
[6] C.Pandu Rangan, K.R.Parthasarathy and V.Prakash, On the g-centroidal problem in special classes of perfect graphs, *Ars Combin.* Vol. 50 (1998), pp.267-278.
[7] P.Veeraraghavan, An efficient g-centroid location algorithm for cographs, *International Journal of Mathematics and Mathematical Sciences*, vol. 9, (2005), pp. 1405-1413.
[8] P.Veeraraghavan, Application of g-convexity in mobile ad hoc networks, *communicated (2008)*

Authors

Prakash Veeraraghavan obtained his PhD from Indian Institute of Technology, Chennai, India. His research interest includes Algorithms, optimization, protocols and security. He is currently working as a senior lecturer in the Department of Computer Science and Computer Engineering, La Trobe University.