International Journal of Multimedia and Ubiquitous Engineering
Vol.11, No.10 (2016), pp.45-56
http://dx.doi.org/10.14257/ijmue.2016.11.10.04

Study on the Control and Projective Synchronization of a New Fractional-order Chaos System

Shao Ke-Yong, Chen Bai-Quan, Wang Ting-Ting, Xu Xiang, Zhang Ting-Ting and Ma Di

College of Electrical Information Engineering, Northeast Petroleum University, Daqing 163318, China
411187747@qq.com

Abstract

In this paper, a new fractional-order chaotic system was studied, and the basic dynamic properties of the new system were analyzed. The control of the new system in unstable equilibrium point was realized by designing the reasonable nonlinear controller. Further, we investigated the projective synchronization problem about fractional order chaotic systems with different dimensions, the appropriate controller was designed to achieve the projective synchronization of a four-dimensional drive system and a three-dimensional response system by reducing the dimension. With the numerical simulation of Matlab, the feasibility of the scheme was verified.

***Keywords**: Fractional-order chaos system, Projective synchronization, Controller, Different dimension*

1. Introduction

In the 1980s, Mandelbort pointed out that the motion system described by fractional-order calculus was very similar to the motion system of the integer order calculus [1]. Since then, people gradually shifted the research to the fractional calculus theory. Fractional order calculus extends the description ability which the familiar integer order calculus does. Fractional order could be said to be the promotion of the integer order, while the integer order is only a special case of fractional order differential equation, so the study of fractional order system is also of more universality [2].

Chaos, as a kind of complex nonlinear dynamic behavior, has been applied in biology, chemistry, engineering, informatics and other fields, especially in secure communications, signal processing, image processing [3]. For chaotic systems, the fractional-order chaotic system has more research value than integer order chaotic system. Due to the characteristics of its high complexity, it could be applied to secure communication field with better security [4]. Chaos control and chaos synchronization had become the important foundation of application in the field [5-6], many scholars also got a lot of valuable achievements through the constantly research. For instance, Z Xu [7] realized the control of fractional-order chaotic system based on Lyapunov stability theory; X Geng and X B Zhang [8] proved the effectiveness of the proposed method for the system control , and synchronization in the paper was verified by using the Lyapunov second method; L L Huang [9] proposed a new theorem to judge whether system had chaotic phenomena or not, based on Lyapunov stability theory, and then applied it to the control and synchronization of fractional-order chaotic system, so as to realize the projective synchronization of the same dimension case structure; Z Y Yan[10] put forward a kind of more generalized projective synchronization, and the drive system and response system had the scale factors Q and S simultaneously, based on it, F D Zhang [11] achieved the synchronization of fractional-order chaotic systems which have different dimensions, but for the choice of scale factor was more trouble.

ISSN: 1975-0080 IJMUE

In this paper, we will study a new three-dimensional fractional-order chaotic system, the chaotic dynamic characteristics of the system would be analyzed, and try to implement chaotic control to make sure of the new system stabilization in unstable equilibrium points. Simultaneously, through the method of dimension reduction, the appropriate controller will be proposed to realize synchronization about a high dimensional fractional order chaotic system and a low dimensional system. Moreover numerical simulations will be used to realize synchronization between the new three dimensional chaotic response system and the four-dimensional hyperchaotic Lorenz drive system, to verify the validity and feasibility of the scheme.

2. The Main Results

Consider the fractional order chaotic system as follows:

$$\begin{cases} D_t^q x = -ax - by + yz \\ D_t^q y = -cx + dy - xz \\ D_t^q z = rz + xy \end{cases} \tag{1}$$

Where a, b, c, d, r are the real constants. When parameters $(a,b,c,d,r) = (35,3,10,17,-5)$, and different values of q are taken, the system presents different features. With $q = 0.83$, we set up the system initial value respectively: $(x_0, y_0, z_0) = (2,1,3)$. The system phase diagram and attractor figure respectively are shown in Figure 1 and Figure 2. And the system is periodically changed at this time, so there is no chaos. With $q = 0.90$, we also set up the system initial value respectively: $(x_0, y_0, z_0) = (2,1,3)$. Then we get the phase diagram and attractor figure respectively are shown in Figure 3, Figure 4. Obviously there is chaos in the system. With $q = 0.98$ and the system initial value $(x_0, y_0, z_0) = (2,1,3)$, we could see more clearly that system is a chaotic state from Figure 5 and Figure 6.

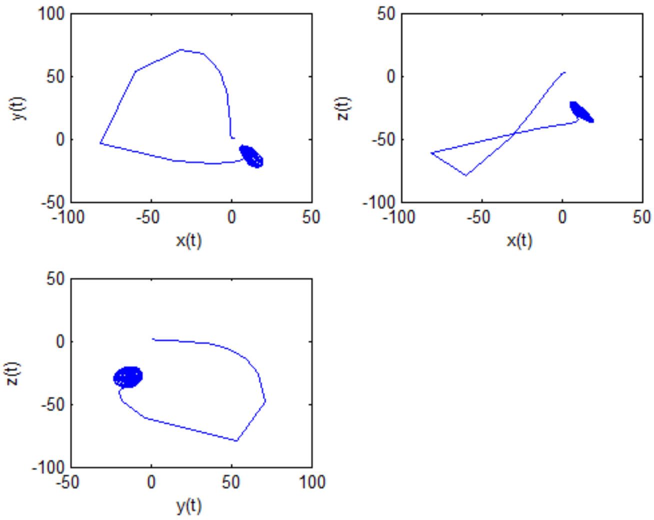

Figure 1. Phase Diagram of System (1) with $q = 0.83$

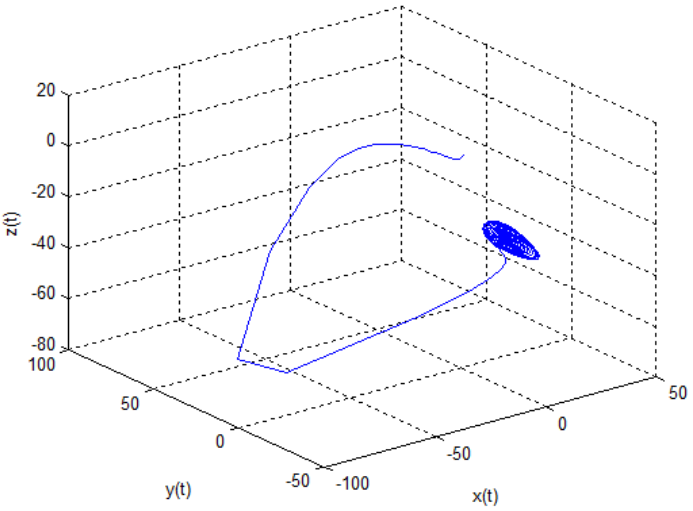

Figure 2. Attractor of System (1) with $q = 0.83$

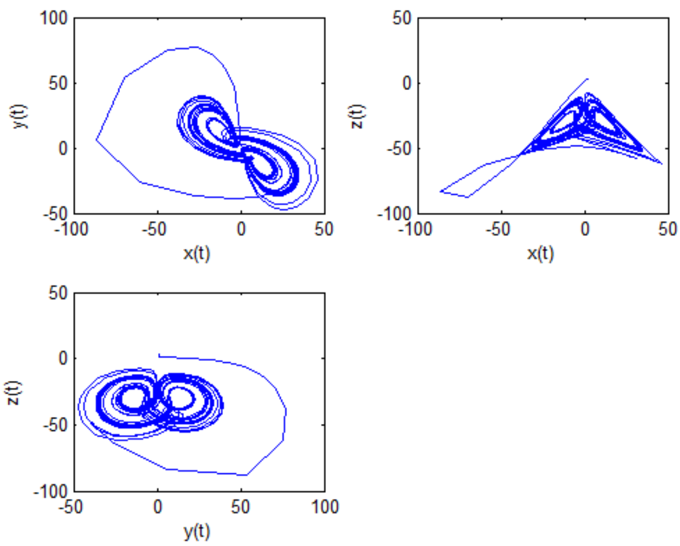

Figure 3. Phase Diagram of System (1) with $q = 0.90$

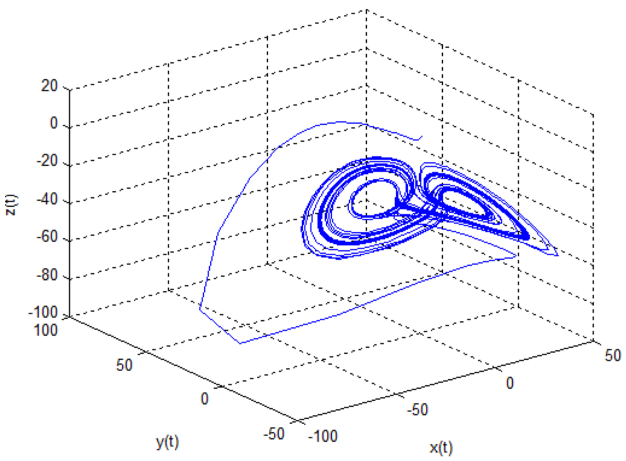

Figure 4. Attractor of System (1) with $q = 0.90$

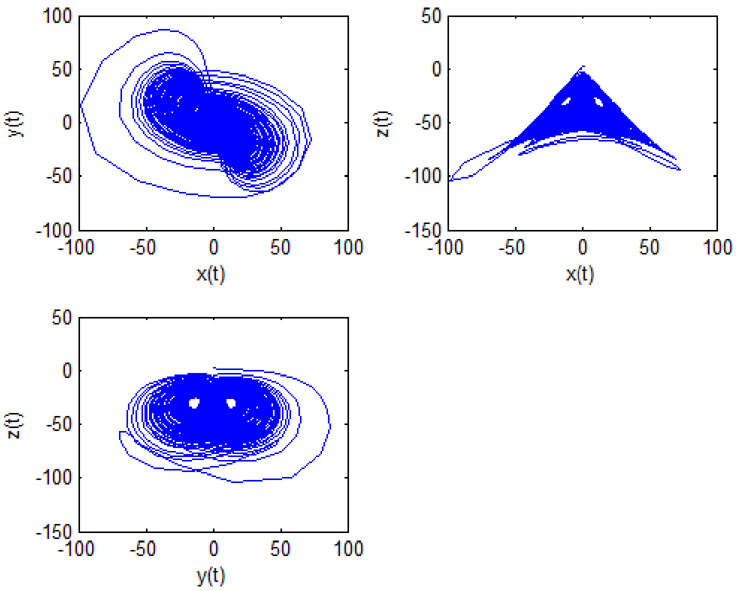

Figure 5. Phase Diagram of System (1) with $q = 0.98$

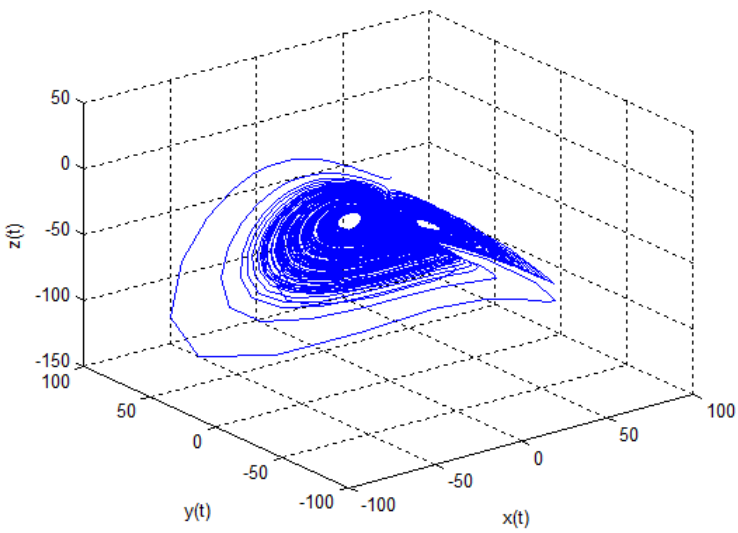

Figure 6. Attractor of System (1) with $q = 0.98$

To calculate the balance point of the new system, equations in system (1) are equal to zero on the left, we can get:

$$\begin{cases} -ax - by + yz = 0 \\ -cx + dy - xz = 0 \\ rz + xy = 0 \end{cases} \qquad (2)$$

When parameters of the equation (2) are taken from $(a, b, c, d, r) = (35, 3, 10, 17, -5)$, the equation (2) could be solved, finally obtaining five equilibrium points of system(1). They are:

$$S_0 = (0,0,0)\ S_1 = (11.4170, -12.5881, -28.7438)$$
$$S_2 = (-11.4170, 12.5881, -28.7438)\ S_3 = (7.6304, 14.2481, 21.7438)$$
$$S_4 = (-7.6304, -14.2481, 21.7438)$$

If the equilibrium point is expressed as a unified expression: $S = (\xi, \tau, \gamma)$, the Jacobian matrix may be expressed as:

$$J = \begin{bmatrix} -35 & -3+\gamma & \tau \\ -10-\gamma & 17 & -\xi \\ \tau & \xi & -5 \end{bmatrix} \quad (3)$$

We may get a different equilibrium in the eigenvalues corresponding Jacobian matrix, when different equilibrium points are substituted into the Jacobian matrix (3).

When the system is at the equilibrium point $S_0 = (0,0,0)$, the obtained eigenvalues are:

$$\lambda_1 = 17.5707, \lambda_2 = -35.5707, \lambda_3 = -5$$

There is a positive characteristic value, indicating that the equilibrium point S_0 is not stable.

When the system is at the equilibrium point $S_1 = (11.4170, -12.5881, -28.7438)$, the obtained eigenvalues are:

$$\lambda_{1,2} = 5.4211 \pm 19.9856i, \lambda_3 = -33.8422$$

Lemma 1[12]: Considering the fractional order systems $D_t^q x = \mathbf{A}x$, where, q is an order number and $0 < q < 1$, $\mathbf{A}$ is a coefficient matrix. If and only if , all eigenvalues of Angle satisfy the condition,

$$\left|\arg(\lambda_i(\mathbf{A}))\right| > \frac{q\pi}{2}, i = 1, 2, \cdots N,$$

The system is asymptotically stable.

According to lemma 1, as long as its eigenvalues meet $\left|\arg(\lambda)\right| > \frac{q\pi}{2}$, the system is asymptotically stable. At this time the eigenvalues of the system of Angle is $\left|\arg(\lambda_1)\right| = \left|\arg(\lambda_2)\right| = 74.8237^\circ$. So it could be deduced the system at the equilibrium point S_1 is stable, when $q < 0.83$. Therefore this may also explain why system (1) does not appear the chaos phenomenon and its orbit periodic change. Thus S_1 is unstable equilibrium point.

The same conclusion may be get easily, from that, S_2, S_3 and S_4 are also unstable.

For system 1, there is

$$\nabla V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -a + d + v = -23 < 0$$

System (1) is a dissipative system because of $\nabla V < 0$.That means all systems of the trajectory of the progressive movement will be fixed in an attractor, finally be restricted to a subset of volume of 0, when $t \to \infty$. It also shows the existence of the attractor.

Then we study the new fractional order chaotic system control problem at the unstable equilibrium points:

In order to make the system (1) stable at the unstable equilibrium point $S_0 = (0,0,0)$, the following controller U is determined as :

$$\begin{cases} u_1 = -k_1 x - yz \\ u_2 = k_2 y + xz \\ u_3 = -k_3 z - xy \end{cases} \tag{4}$$

Under the control of the controller (4), the new fractional order chaotic system is changed as:

$$\begin{cases} D_t^q x = -ax - by + yz + u_1 \\ D_t^q y = -cx + dy - xz + u_2 \\ D_t^q z = rz + xy + u_3 \end{cases}$$

Namely, it equals to this:

$$\begin{cases} D_t^q x = -(a + k_1)x + by \\ D_t^q y = -cx + (d + k_2)y \\ D_t^q z = (r - k_3)z \end{cases} \tag{5}$$

To make Laplace transform on both sides of the equation (5), we let

$$\begin{cases} E_1(s) = L(x(t)) \\ E_2(s) = L(y(t)) \\ E_3(s) = L(z(t)) \end{cases}$$

According to

$$\begin{cases} L(D_t^q x) = s^q E_1(s) - s^{q-1} x(0) \\ L(D_t^q y) = s^q E_2(s) - s^{q-1} y(0) \\ L(D_t^q z) = s^q E_3(s) - s^{q-1} z(0) \end{cases}$$

We have

$$\begin{cases} s^q E_1(s) - s^{q-1} x(0) = -(35 + k_1)E_1(s) + 3E_2(s) \\ s^q E_2(s) - s^{q-1} y(0) = -10E_1(s) + (17 + k_2)E_2(s) \\ s^q E_3(s) - s^{q-1} z(0) = -(5 + k_3)E_3(s) \end{cases} \tag{6}$$

Then solving the equation (6), we have

$$\begin{cases} E_1(s) = \dfrac{s^{q-1} x(0) + 3E_2(s)}{s^q + 35 + k_1} \\ E_2(s) = \dfrac{s^{q-1} y(0) + 10E_1(s)}{s^q - 17 - k_2} \\ E_3(s) = \dfrac{s^{q-1} z(0)}{s^q + 5 + k_3} \end{cases}$$

According to Laplace's final value theorem, we may get

$$\begin{cases} \lim_{t\to\infty} x(t) = \lim_{s\to 0^+} sE_1(s) = \dfrac{s^{q_1}x(0)+3sE_2(s)}{s^{q_1}+35+k_1} \\ \lim_{t\to\infty} y(t) = \lim_{s\to 0^+} sE_2(s) = \dfrac{s^{q_2}y(0)+10sE_1(s)}{s^{q_2}-17-k_2} \\ \lim_{t\to\infty} z(t) = \lim_{s\to 0^+} sE_3(s) = \dfrac{s^{q_3}z(0)}{s^{q_3}+5+k_3} \end{cases}$$

When $k_1 > -35, k_2 < -17, k_3 > -5$, we may infer that $\begin{cases} \lim_{t\to\infty} x(t) = 0 \\ \lim_{t\to\infty} y(t) = 0 \\ \lim_{t\to\infty} z(t) = 0 \end{cases}$.

So, the fractional order system under the controller(4) asymptotically stable at the equilibrium point $S_0 = (0,0,0)$ with $t \to \infty$.

We set up the system initial value x_0, y_0, z_0, the order q, the gain k_1, k_2, k_3 respectively:

$(x_0, y_0, z_0) = (-2,1,3)$, $q = 0.98$ and $(k_1, k_2, k_3) = (10,-20,0)$.

In the light of the prediction-correction method[13], the simulation results are shown in Figure 7.

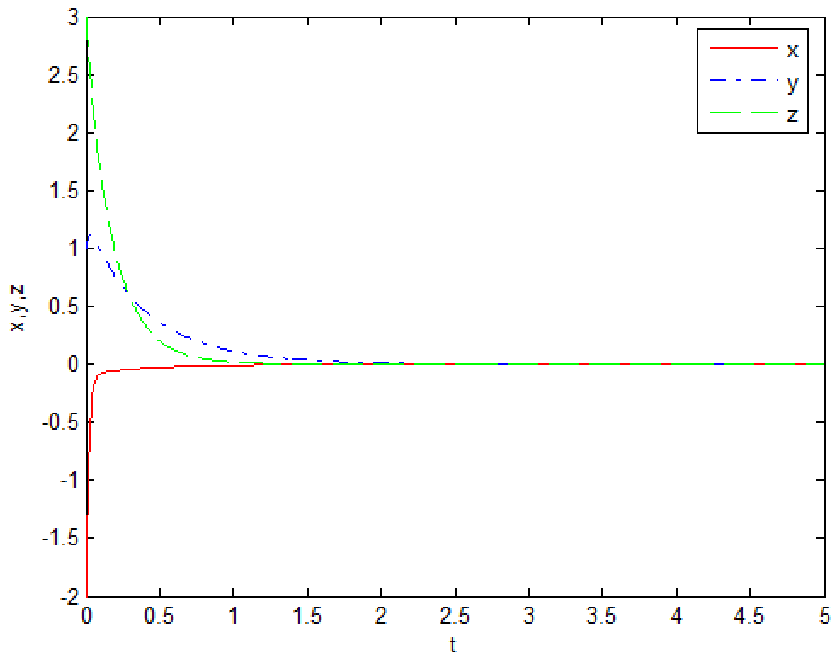

Figure 7. The Stability Curve of x, y, z, when System(1) is Controlled

We could see, from figure 7, the system quickly tends to zero in a short time after adding a controller (4). And the results have proved the feasibility of the controller (4).

Based on the analysis of dynamic characteristics of a new fractional order chaotic systems, we continue to research the projection synchronization problem of fractional-order chaotic systems with different dimensions.

For researching the projection synchronization problem about fractional-order chaotic systems with different dimensions, now we select the four-dimensional fractional order hyperchaos Lorenz system as the drive system , and the system model is described as follows:

$$\begin{cases} D_t^q x_1 = 10(x_2 - x_1) + x_4 \\ D_t^q x_2 = 28x_1 - x_2 - x_1 x_3 \\ D_t^q x_3 = -8x_3 / 3 + x_1 x_2 \\ D_t^q x_4 = -x_4 - x_2 x_3 \end{cases} \tag{7}$$

When $q = 0.98$, the system is in a chaotic state.

Then we choose the new fractional order three-dimensional chaotic system (1) as the response system. When $q = 0.98$, there is chaos in the system. After generation into the system of the parameters, the system is as the following form:

$$\begin{cases} D_t^q y_1 = -35y_1 - 3y_2 + y_2 y_3 + u_1 \\ D_t^q y_2 = -10y_1 + 17y_2 - y_1 y_3 + u_2 \\ D_t^q y_3 = -5y_3 + y_1 y_2 + u_3 \end{cases} \tag{8}$$

For any fractional order drive system and the response system could be written as follows:

$$D_t^q x = \mathbf{A}x + F(x) \tag{9}$$

$$D_t^q y = \mathbf{B}y + G(y) + U \tag{10}$$

Where $0 < q < 1$, matrix A and B , respectively, are the linear parts of the drive system and response system, $F(x)$ and $G(y)$ are the nonlinear parts of the two systems. U is the controller to be determined later.

Define 1[14]: For the given drive system and response system, synchronization error variable e is defined as: $e = y - Sx$. If there is a suitable controller U, making

$$\lim_{t\to\infty}\|e\| = \lim_{t\to\infty}\|y - Sx\| = 0$$

Where $S \in R^{n\times m}$, $e = (e_1, e_2, \cdots e_n)^T$ .It can prove that the projective synchronization has been realized between drive system and response system.

Theorem 1: Design the controller U of drive system and response system as:

$$U = S\mathbf{A}x - \mathbf{B}Sx + SF(x) - G(y) - V(t)$$

Where $V(t) = \mathbf{P}e$, if $|\arg(\lambda(\mathbf{B} - \mathbf{P}))| > \frac{q\pi}{2}$ is established, then the error system is stability, so the projective synchronization of the system (9) and the system (10) is achieved.

Prove:

According to Define 1 synchronization error of system (9) and system (10) is defined as follows:

$$\begin{aligned}D_t^q e &= D_t^q y - SD_t^q x\\ &= \mathbf{B}y + G(y) + S\mathbf{A}x - \mathbf{B}Sx + SF(x) - G(y) - V(t) - S(\mathbf{A}x + F(x))\\ &= \mathbf{B}y - S\mathbf{B}x - \mathbf{P}e\\ &= \mathbf{B}(y - Sx) - \mathbf{P}e\\ &= \mathbf{B}e - \mathbf{P}e\\ &= (\mathbf{B} - \mathbf{P})e\end{aligned} \quad (11)$$

From Lemma 1, just choosing the appropriate matrix $\mathbf{P}$, so as to satisfy the condition $\left|\arg(\lambda(\mathbf{B}-\mathbf{P})\right| > \frac{q\pi}{2}$, then the error system (11) is asymptotically stable, namely the establishment of $\lim_{t\to\infty}\|e\| = 0$. Therefore the projective synchronization of the system (9) and system (10) is realized.

Considering the synchronization problem for systems (7) and systems (8).

When $S = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}$, synchronization error is $\begin{cases} e_1 = y_1 - x_1\\ e_2 = y_2 - x_2\\ e_3 = y_3 - x_3\end{cases}$.

According to the previous form of equation (9) and (10). Matrix $\mathbf{B}$ can be obtained as

$$B = \begin{bmatrix} -35 & -3 & 0\\ -10 & 17 & 0\\ 0 & 0 & -5\end{bmatrix}$$

Matrix $\mathbf{P}$ is taken as $\mathbf{P} = \begin{bmatrix} -35 + p_1 & -3 & 0\\ -10 & 17 + p_2 & 0\\ 0 & 0 & -5 + p_3\end{bmatrix}$, where $p_1, p_2, p_3 > 0$.

When $q = 0.98$, the error system $D_t^q e = (\mathbf{B} - \mathbf{P})e$, is equal to

$$D_t^q e = \begin{bmatrix} -p_1 & 0 & 0\\ 0 & -p_2 & 0\\ 0 & 0 & -p_3\end{bmatrix}\begin{bmatrix} e_1 & e_2 & e_3\end{bmatrix}^T$$

Obviously, none of the eigenvalues of the matrix $\mathbf{P}$ is greater than zero, which satisfies $\left|\arg(\lambda(\mathbf{B}-\mathbf{P})\right| > \frac{q\pi}{2}$.Therefore, system (7) and system (8) achieve the synchronization.

We respectively set the initial values of drive system (7) and response system (8) are: $(x_1(0), x_2(0), x_3(0), x_4(0)) = (1, -1, 1, 1)$ and $(y_1(0), y_2(0), y_3(0)) = (-5, 3, -2)$.

Meanwhile, we let $(p_1, p_2, p_3) = (1, 2, 3)$. The simulation results are shown in Figure 8.

From Figure 8, we can see e_1, e_2 and e_3 are asymptotically stable to the origin after a relatively short period of time. As a result, this also shows that synchronization of the two systems has been achieved.

3. Conclusion

In summary, we investigated a new three-dimensional fractional-order chaotic system which had five parameters. By observing the system phase diagram and the attractor figure in different order, we have judged whether the system appeared chaotic phenomenon. The characteristic of the new fractional order chaotic system has been analyzed from the stability of the system's equilibrium points and the dissipative. Theoretically it proved that the new system was chaotic. Then the control of the new chaotic system has been realized by designing proper controller based on Laplace transform. Numerical simulations were used to verify the effectiveness of the proposed controller .In the last, we took the new system as an example to achieve the projective synchronization between a high dimension fractional order chaotic system and a low dimension fractional order chaotic system by reducing the dimension. Compared with other synchronizations, the key to achieve synchronization is how to choose the projection factor S and how to set up the value of matrix $\mathbf{P}$. With the numerical simulation of Matlab, the results show that the synchronization of the two systems has been achieved in a short period of time.

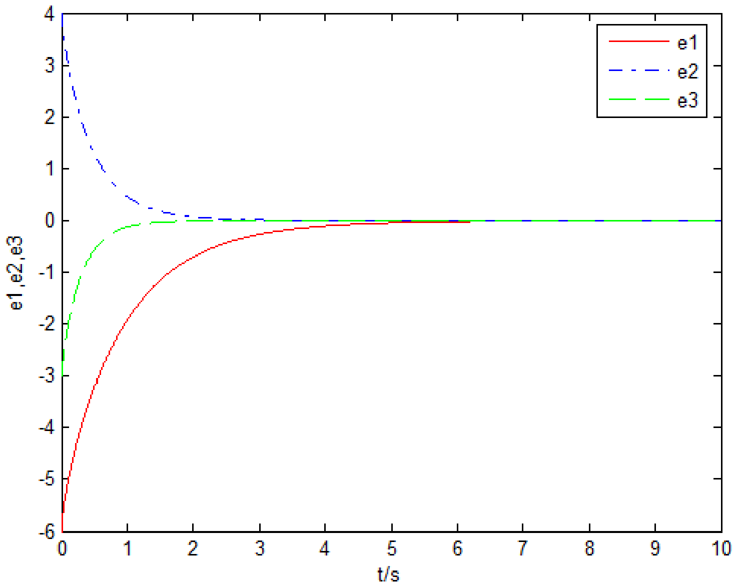

Figure 8. Synchronization Error Curve of System (7) and (8)

Acknowledgements

This work is supported by Northeast Petroleum University Innovation Foundation For Postgraduate(YJSCX2015-031NEPU), the National Natural Science Foundation of China (51404073), the Scientific Research Fund of Heilongjiang Provincial Department of Education (12541090) and Youth Science Foundation of Northeast Petroleum University (2013NQ105).

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Authors

Shao Ke-Yong, (1970), male, he received the Ph.D. degree from Northeastern University, major in Control Theory and Control Engineering. He is the professor of Northeast Petroleum University now. He is mainly engaged in teaching and scientific research work including intelligent control, chaos control, the robust control of complex systems theory.

Chen Bai-Quan, (1990), male, he is a postgraduate, major in Control Theory and Control Engineering. He is now studying in Northeast Petroleum University in China. He is mainly engaged in the research of chaos control and nonlinear control.