International Journal of Multimedia and Ubiquitous Engineering
Vol. 10, No. 2 (2015), pp. 111-120
http://dx.doi.org/10.14257/ijmue.2015.10.2.10

Hopf Bifurcation Analysis in a Delayed Ratio-dependent Gause-type Predator-prey Models

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Abstract

A class of delayed ratio-dependent Gause-type predator-prey model is considered. Firstly, we study the eigenvalue problem for the linearized system at the coexisting equilibrium and a group of sufficient conditions for the existence of Hopf bifurcation are obtained. Secondly, the direction of the Hopf bifurcation and the stability of the bifurcated periodic solutions are determined by applying the normal form method and the center manifold theorem. Finally, some numerical simulations are carried out to illustrate the obtained results.

Keywords: *Gause-type model, Ratio-dependent, Hopf bifurcation*

1. Introduction

Not only in pure ecology, but also in mathematical biology, the importance of the relationship between predator and prey will continue to be widely studied [1-4]. It has attracted extensive studies on the dynamics of various multi-species predator-prey models. In the predator-prey model, Holling II response functions is what we say Michaelis-Menten type. The dynamics of models have been studied extensively. See [5-11].

In the study of predator-prey interactions with Michaelis-Menten type was the discovery of the well known`` paradox of enrichment "[7, 9]. Arditi and Ginzburg [3] put forward the following ratio dependent functional response: $p\left(\frac{x}{y}\right)=\frac{cx/y}{m+x/y}$, and studyed the predator-prey system with this functional response. Beretta and Kuang [5] gived some preliminary results and provided sufficient conditions for the positive equilibrium to be globally asymptotically by construct a Lyapunov function. In 2001, Xiao and Ruan [8] studied the qualitative properties near the origin and proved $O(0,0)$ is a critical point. There are several different topological structures around it, including parabola, ellipse and the hyperbolic orbit. The existence conditions of limit cycle are given by numerical simulation.

In nature, the population will generally experience some periodic oscillation. It can be regarded as by the impact of delay from a mathematical point. See [12-17]. In 1998, Beretta and Kuang lead a single discrete time delay to predator equation of the following model:

$$\begin{cases} \dot{x} = ax\left(1-\dfrac{x}{K}\right)-\dfrac{cxy}{my+x} \\ \dot{y} = y\left(-d+\dfrac{fx(t-\tau)}{my(t-\tau)+x(t-\tau)}\right) \end{cases} \tag{1.1}$$

For this model, the authors studyed the stability of periodic solutions and the existence of

ISSN: 1975-0080 IJMUE

Hopf bifurcation in [18]. In order to prove the global asymptotic stability of the positive equilibrium, some sufficient conditions are given by constructing appropriate Lyapunov functions.

The purpose of current work is to analysis the effect of delay on the dynamics for the following delay ratio-dependent food chain model:

$$\begin{cases} \dot{x} = ax\left(1 - \dfrac{x}{K}\right) - \dfrac{cxy}{my + x} \\ \dot{y} = y\left(-d + \dfrac{fx(t-\tau)}{my(t-\tau) + x(t-\tau)} - rz\right) \\ \dot{z} = z(-s + ey(t-\tau)) \end{cases} \qquad (1.2)$$

Here z is top predator. a, K, c, m, d, f are positive constants whose biological meaning are obvious. r, s, e are positive constants that stand for capturing rate, half capturing saturation constant, conversion rate, top-predator death rate, respectively. In [19], the authors reveal that Hopf bifurcation can occur as the delay crosses some critical values which lead to the existence of periodic solution that may conform to certain phenomena in ecosystem system. We still let τ as the bifurcation parameter in this paper and consider the delay Gause-type predator-prey model with ratio-dependent functional response.

2. Stability and Hopf Bifurcation of Coexisting Equilibrium

For the sake of convenience, we non-dimensionalizes the Eq.(1.2), then the Eq.(1.2) takes the form:

$$\begin{cases} \dot{x} = x(1 - x) - \dfrac{pxy}{y + x} \\ \dot{y} = y\left(-l + \dfrac{qx(t-\tau)}{y(t-\tau) + x(t-\tau)} - z\right) \\ \dot{z} = z(-s + uy(t-\tau)) \end{cases} \qquad (2.1)$$

which satisfies $x_0(\theta) = \phi_1(\theta) \geq 0$, $y_0(\theta) = \phi_2(\theta) \geq 0$, $z_0(\theta) = \phi_3(\theta) \geq 0$, $\theta \in [-\tau, 0]$, $x(0) > 0$, $y(0) > 0$, $z(0) > 0$, $\phi(\theta) = (\phi_1, \phi_2, \phi_3) \in C([-1, 0], R^3)$,

$\|\phi\| = \max\{|\phi(\theta)| : \theta \in [-\tau, 0]\}$,and $|\phi|$ is any norm in R^3.Where

$$p = \frac{c}{ma}, q = \frac{f}{a}, l = \frac{d}{a}, u = \frac{ek}{m}.$$

Obviously, the delay can't change the number of equilibria and non-dimensionalizes can't change the properties of system. In the following, we always assume Eq.(2.1) has a positive equilibrium exists and denote it by $\overline{E}(x^*, y^*, z^*)$ with

$$x^* = \frac{(s/u-1) \pm \sqrt{(s/u-1)^2 + 4s(p-1)/u}}{2}, \; y^* = \frac{s}{u}, \quad z^* = \frac{qx^*}{x^*+y^*} - l$$

We consider the linearized system of (2.1) at $\overline{E}$. The characteristic equation at $\overline{E}$ is given by

$$D(\lambda,\tau) = \lambda^3 + a_2\lambda^2 + (b_2\lambda^2 + b_1\lambda + b_0)e^{-\lambda\tau} = 0, \qquad (2.2)$$

Where $a_2 = -m_{11}, b_2 = -n_{22} > 0, b_1 = m_{11}n_{22} - m_{12}n_{21} - m_{23}n_{32}$ and $b_0 = m_{11}m_{23}n_{32}$.

$$m_{11} = 1 - 2x^* - \frac{py^{*2}}{(x^*+y^*)^2}, \quad m_{12} = -\frac{px^{*2}}{(x^*+y^*)^2} < 0, \quad m_{23} = -y^* < 0,$$

$$n_{21} = \frac{qy^{*2}}{(x^*+y^*)^2} > 0, \quad n_{22} = -\frac{qx^*y^*}{(x^*+y^*)^2} < 0, \quad n_{32} = uz^* > 0.$$

If $m_{11} < 0$,then all the eigenvalues of Eq.(2.2) have negative real parts when $\tau = 0$ by the Routh-Hurwitz criterion. Now we substitute $\lambda = i\omega$ ($\omega > 0$) into Eq.(2.2),

(1) when $\omega = 0$, $D(0,\tau) = b_0 = m_{11}m_{23}n_{32} \neq 0$;

(2) when $\omega \neq 0$, $D(i\omega,\tau) = (i\omega)^3 + a_2(i\omega)^2 + [b_2(i\omega)^2 + b_1 i\omega + b_0]e^{-i\omega\tau} = 0$.

Separating the real and imaginary parts gives

$$-a_2\omega^2 - b_2\omega^2\cos\omega\tau + b_1\omega\sin\omega\tau + b_0\cos\omega\tau = 0 \qquad (2.3)$$

and

$$-\omega^3 + b_2\omega^2\sin\omega\tau + b_1\omega\cos\omega\tau - b_0\sin\omega\tau = 0 \qquad (2.4)$$

We get

$$\omega^6 + (a_2^2 - b_2^2)\omega^4 + (2b_0b_2 - b_1^2)\omega^2 - b_0^2 = 0 \qquad (2.5)$$

Let $\omega^2 = l$, $P_1 = a_2^2 - b_2^2$, $P_2 = 2b_0b_2 - b_1^2$ and $P_3 = -b_0^2 < 0$,then Eq.(2.5) become

$$l^3 + P_1l^2 + P_2l + P_3 = 0 \qquad (2.6)$$

From Ruan and Wei [20], we have the following results on the distribution of roots of Eq.(2.6).

Denote $h(l) = l^3 + P_1l^2 + P_2l + P_3$, $l_1 = \frac{-P_1 + \sqrt{P_1^2 - 3P_2}}{3}$ and $l_2 = \frac{-P_1 - \sqrt{P_1^2 - 3P_2}}{3}$

Because $P_3 = -b_0^2 < 0$,then we have the following Lemma:

Lemma 2.1. When $P_1^2 - 3P_2 > 0$,

(a) $h(l_1) > 0, h(l_2) < 0$,then Eq. (2.6) has one positive root, this implies that the characteristic Eq.(2.2) has only a pair of purely imaginary roots.

(b) $h(l_1) < 0, h(l_2) > 0$,then Eq. (2.6) has three positive roots, This implies that the haracteristic Eq.(2.2) has three pairs of purely imaginary roots.

Without loss of generality, we assume that it has three positive roots, denoted by $l_1, l_2\ l_3$, respectively. Then Eq.(2.2) has three positive roots $\omega_i = \sqrt{l_i}$ (i=1,2,3). We can calculate

$$\tau_{\mathrm{i}} = \frac{1}{\omega_{\mathrm{i}}}\left\{ \arccos \frac{b_1\omega_{\mathrm{i}}^4 + (b_2\omega_0^2 - b_0)(-a_2\omega_{\mathrm{i}}^2)}{(b_2\omega_{\mathrm{i}}^2 - b_0)^2 + b_1\omega_{\mathrm{i}}^2} \right\},\quad \text{i=1,2,3}$$

Define

$$\tau_0 = \tau_{\mathrm{k}} = \min\{\tau_i\}, \qquad \omega_0 = \omega_k, \tag{2.7}$$

Denoting $\lambda(\tau) = \alpha(\tau) + \beta(\tau)$ be the root of Eq.(2.2) satisfying $\alpha(\tau_0) = 0$, $\beta(\tau_0) = \omega_0$, we have the following lemma.

Lemma 2.2. If $h'(\omega_0^2) \neq 0$, we have $\frac{\mathrm{d}\,\mathrm{Re}\,\lambda(\tau)}{\mathrm{d}\tau} > 0$.

Proof. Assume

$$h(\omega_0^2) = \omega_0^6 + (a_2^2 - b_2^2)\omega_0^4 + (2b_0b_2 - b_1^2)\omega_0^2 - b_0^2 = 0,$$

Because

$$sign\left[\frac{\mathrm{d}\,\mathrm{Re}\,\lambda(\tau)}{\mathrm{d}\tau}\right]\Bigg|_{\tau=\tau_0} = sign\left[\frac{\mathrm{d}\,\mathrm{Re}\,\lambda(\tau)}{\mathrm{d}\tau}\right]^{-1}\Bigg|_{\tau=\tau_0},$$

$$\left[\frac{\mathrm{d}\,\mathrm{Re}\,\lambda(\tau)}{\mathrm{d}\tau}\right]^{-1} = \mathrm{Re}\left[\frac{(3\lambda^2 + 2a_2\lambda)e^{\lambda\tau}}{\lambda(\mathrm{b}_2\lambda^2 + \mathrm{b}_1\lambda + \mathrm{b}_0)}\right]\Bigg|_{\tau=\tau_0} + \mathrm{Re}\left[\frac{2b_2\lambda + b_1}{\lambda(\mathrm{b}_2\lambda^2 + \mathrm{b}_1\lambda + \mathrm{b}_0)}\right]\Bigg|_{\tau=\tau_0}$$

$$= \frac{3\omega_0^4 + 2(a_2^2 - b_2^2)\omega_0^2 + (2b_0b_2 - b_1^2)}{b_1^2\omega_0^4 + (b_0 - b_2\omega_0^2)^2\omega_0^2}$$

$$= \frac{h'(\omega_0^2)}{b_1^2\omega_0^4 + (b_0 - b_2\omega_0^2)^2\omega_0^2}$$

We have

$$sign\left[\frac{\mathrm{d}\,\mathrm{Re}\,\lambda(\tau)}{\mathrm{d}\tau}\right]^{-1}\Bigg|_{\tau=\tau_0} = sign\ (h'(\omega_0^2)).$$

If $h'(\omega_0^2) \neq 0$, then $\frac{\mathrm{d}\,\mathrm{Re}\,\lambda(\tau)}{\mathrm{d}\tau} \neq 0$. There must be $\frac{\mathrm{d}\,\mathrm{Re}\,\lambda(\tau)}{\mathrm{d}\tau} > 0$. This is because Eq.(2.2) has the positive real part roots as $\tau < \tau_0$ if $\frac{\mathrm{d}\,\mathrm{Re}\,\lambda(\tau)}{\mathrm{d}\tau}\Big|_{\tau=\tau_0} < 0$. This contradicts to the fact when $\tau \in [0, \tau_0)$ and $\overline{E}(x*,\ y*,\ z*)$ is asymptotically stable.

By Lemma 2.2 we have the following theorem.

Theorem 2.1. Suppose that $m_{11} < 0$, If Lemma 2.1 holds, then the equilibrium

$\overline{E}(x^*, y^*, z^*)$
of the delay model (1.2) is asymptotically stable when $\tau < \tau_0$,and unstable when $\tau > \tau_0$,where τ_0 is de_ned by (2.7). In addition,if $h'(\omega_0^2) \neq 0$,then Hopf bifurcation occurs when $\tau = \tau_0$.

3. Direction and Stability of Hopf Bifurcation

Let $x_1(t) = x(t) - x^*, x_2(t) = y(t) - y^*, x_3(t) = z(t) - z^*, X_i(t) = x_i(\tau t)$ (i=1,2,3), $\tau = \tau_0 + \mu, \mu \in R$ and

$$B_1 = \begin{pmatrix} m_{11} & m_{12} & 0 \\ 0 & 0 & m_{23} \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ n_{21} & n_{22} & 0 \\ 0 & n_{32} & 0 \end{pmatrix}$$

The system (1.2) is transformed into a functional di_erential equation (FDE) in $C = C([-1, 0], R^3)$, defining

$$L_\mu(\phi) = (\tau_0 + \mu)B_1\phi(0) + (\tau_0 + \mu)B_2\phi(-1), \qquad (3.1)$$

Where $\phi = (\phi_1, \phi_2, \phi_3)^T \in C([-1, 0], R^3)$. And the nonlinear term is

$$f(\mu, \phi) = (\tau_0 + \mu)\begin{pmatrix} g_{12}\phi_1(0)\phi_2(0) + g_{13}\phi_1^2(0) + g_{14}\phi_2^2(0) + \cdots\cdots \\ -\phi_2(0)\phi_3(0) + g_{22}\phi_2(0)\phi_1(-1) + g_{23}\phi_2(0)\phi_2(-1) + g_{24}\phi_1(-1)\phi_2(-1) \\ + g_{25}\phi_1^2(-1) + g_{26}\phi_2^2(-1) + \cdots\cdots \\ u\phi_3(0)\phi_2(-1) + \cdots\cdots \end{pmatrix}$$

Where $g_{12} = -\frac{2px^*y^*}{(x^*+y^*)^3}, g_{13} = \frac{py^{*2}}{(x^*+y^*)^3} - 1, g_{14} = \frac{px^{*2}}{(x^*+y^*)^3}, g_{22} = \frac{qy^*}{(x^*+y^*)^2}$,
$g_{23} = -\frac{qx^*}{(x^*+y^*)^2}, g_{24} = \frac{q(x^*-y^*)}{(x^*+y^*)^3}, g_{25} = -\frac{qy^*}{(x^*+y^*)^3}, g_{26} = \frac{qx^*y^*}{(x^*+y^*)^3}$.

Obviously $\mu = 0$ is a Hopf bifurcation point. So the system (3.1) can transform into an abstract functional differential equation:

$$\dot{v}(t) = L_\mu(v_t) + f(\mu, v_t) \quad , \qquad (3.2)$$

Where $v(t) = (x_1(t), x_2(t), x_3(t))^T \in R^3$.

There exists a 3×3 matrix $\eta(\theta, \mu)$ $(-1 \le \theta \le 0)$, whose elements are of bounded variation functions such that

$$L_\mu(\phi) = \int_{-1}^{0}[\mathrm{d}\,\eta(\theta, \mu)]\phi(\theta), \quad \text{for } \phi \in C([-1, 0], R^3). \qquad (3.3)$$

In fact, we can choose

$$\eta(\theta,\mu)=\begin{cases}(\tau_0+\mu)\ B_1, & \theta=0\\ 0, & \theta\in(-1,0)\\ (\tau_0+\mu)\ B_1, & \theta=-1\end{cases} \tag{3.4}$$

Then Eq.(3.3) is satisfied. For $\phi\in C([-1,0],\ R^3)$, we define

$$A(\mu)\phi(\theta)=\begin{cases}\dfrac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}, & \theta\in[-1,0),\\ \int_{-1}^{0}[\mathrm{d}\eta(\xi,\mu)]\phi(\xi), & \theta=0\end{cases} \tag{3.5}$$

and

$$R(\mu)\phi(\theta)=\begin{cases}0, & \theta\in[-1,0),\\ f(\mu,\phi), & \theta=0\end{cases} \tag{3.6}$$

So Eq.(3.3) is equivalent to the following abstract equation

$$\dot{x}_t=A(\mu)x_t+R(\mu)x_t, \tag{3.7}$$

Where $x=(x_1,x_2,x_3)^T$. and $x_t(t)=x(t+\theta)$ for $\theta\in[-1,0]$.
For $\psi\in C^1([0,1],\ R^{3*})$, we define

$$A^*\psi(s)=\begin{cases}-\dfrac{\mathrm{d}\psi(s)}{\mathrm{d}s}, & s\in[-1,0),\\ \int_{-1}^{0}\psi(-\xi)\mathrm{d}\eta(\xi,0), & s=0\end{cases} \tag{3.8}$$

and a bilinear form

$$\langle\psi(s),\phi(\theta)\rangle=\overline{\psi}(0)\phi(0)-\int_{-1}^{0}\int_{\xi=0}^{\theta}\overline{\psi}(\xi-\theta)\mathrm{d}\eta(\theta)\phi(\xi)\mathrm{d}\xi,$$

where $\eta(\theta)=\eta(\theta,0)$ .Then $A(0)$ and A^* are adjoint operators. We know that $\pm i\omega_0\tau_0$ are eigen values of A(0) and therefore they are also eigenvalues of A*(0). The vectors $q(\theta)=(q_1,1,q_3)^T e^{i\omega_0\tau_0}$ $(\theta\in[-1,0])$ and $q^*(s)=D(q_1^*,1,q_3^*)e^{i\omega_0\tau_0 s}$ $(s\in[-1,0])$ are the eigenvectors of A(0) and A* corresponding to the eigenvalue $i\omega_0\tau_0$ and $-i\omega_0\tau_0$, respectively, satisfying $\langle q^*(s),q(\theta)\rangle=1$, $\langle q^*(s),\overline{q}(\theta)\rangle=0$ with

$q_1=\dfrac{m_{12}\tau_0}{i\omega_0-m_{11}\tau_0}$, $q_3=\dfrac{n_{32}\tau_0 e^{-i\omega_0}}{i\omega_0}$, $q_1^*=\dfrac{n_{21}\tau_0 e^{-i\omega_0}}{i\omega_0-m_{11}\tau_0}$, $q_3^*=-\dfrac{m_{23}\tau_0}{i\omega_0}$ and

$\overline{D}=\dfrac{1}{q_1^*\overline{q}_1+1+q_3^*\overline{q}_3+(n_{21}\overline{q}_1+n_{22}+n_{32}q_3^*)\tau_0 e^{i\omega\ \tau_0}}$ .Following the same algorithms as Hassard, Kazarinoff and Wan [21], we can obtain the coefficients which will be used to determine the important quantities:

$$g_{02}=2\overline{D}\tau_0[g_{12}\overline{q_1 q_1^*}+g_{13}\overline{q_1^*}\,\overline{q_1}^2+g_{14}\overline{q_1^*}-2\overline{q_3}+(g_{22}\overline{q_1}+g_{23}+2u\overline{q_3^*}\,\overline{q_3})e^{i\omega\ \tau_0}+(g_{24}\overline{q_1}$$

$$+ g_{25}\overline{q_1}^2 + g_{26})e^{2i\omega\tau_0}],$$

$$g_{20} = 2\overline{D}\tau_0[g_{12}q_1\overline{q_1^*} + g_{13}\overline{q_1^*}q_1^2 + g_{14}\overline{q_1^*} - 2q_3 + (g_{22}q_1 + g_{23} + u\overline{q_3^*}q_3)e^{-i\omega\tau_0} + (g_{24}q_1$$
$$+ g_{25}q_1^2 + g_{26})e^{-2i\omega\tau_0}],$$

$$g_{11} = 2\overline{D}\tau_0[g_{12}\,\mathrm{Re}\{q_1\}\overline{q_1^*} + g_{13}q_1\overline{q_1}\overline{q_1^*} + g_{14}\overline{q_1^*} - 2\,\mathrm{Re}\{q_3\} + (g_{22}\,\mathrm{Re}\{q_1e^{-i\omega\tau_0}\}$$
$$+ g_{23}\,\mathrm{Re}\{e^{-i\omega\tau_0}\} + g_{24}\,\mathrm{Re}\{q_1\} + g_{25}q_1\overline{q_1} + g_{26} + 2u\,\mathrm{Re}\{q_3e^{i\omega\tau_0}q_3^*\}],$$

$$g_{21} = \overline{D}\tau_0[(g_{12}q_1\overline{q_1^*} + 2g_{14}\overline{q_1^*} - 2q_3 + g_{23}e^{-i\omega\tau_0})W_{11}^2(0) + (\frac{g_{12}}{2}\overline{q_1}\overline{q_1^*} + g_{14}\overline{q_1^*} - \overline{q_3}$$
$$+ \frac{g_{22}}{2}\overline{q_1}e^{i\omega_0\tau_0} + \frac{g_{23}}{2}e^{i\omega_0\tau_0})W_{20}^2(0) + (\frac{g_{12}}{2}\overline{q_1^*} + g_{13}\overline{q_1^*}\overline{q_1})W_{20}^1(0) + (g_{12}\overline{q_1^*} + 2g_{13}\overline{q_1^*}q_1)$$
$$W_{11}^1(0)) + (2ue^{-i\omega_0\tau_0}\overline{q_3} - 2)W_{11}^3(0) + (ue^{i\omega_0\tau_0}\overline{q_3^*} - 1)W_{20}^3(0) + (g_{23} + 2g_{26}e^{-i\omega_0\tau_0} + 2u\overline{q_3^*}q_3 +$$
$$\frac{g_{24}}{2}q_1e^{-i\omega_0\tau_0})W_{11}^2(-1) + (\frac{g_{23}}{2} + g_{26}e^{i\omega_0\tau_0} + u\overline{q_3}\overline{q_3^*} + \frac{g_{24}}{2}\overline{q_1}e^{i\omega_0\tau_0})W_{20}^2(-1) + (g_{22}$$
$$+ 2g_{25}q_1e^{-i\omega_0\tau_0}\frac{g_{24}}{2}e^{-i\omega_0\tau_0}) + (\frac{g_{22}}{2} + g_{26}\overline{q_1}e^{-i\omega_0\tau_0} + u\overline{q_3}\overline{q_3^*} + \frac{g_{24}}{2}e^{i\omega_0\tau_0})W_{20}^1(-1)]$$

Since there are $W_{20}(\theta)$ and $W_{11}(\theta)$ in g_{21}, we still need to compute them. From [21], we have：

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_0}q(\theta) + \frac{i\overline{g_{02}}}{3\omega_0\tau_0}\overline{q}(\theta) + E_1e^{2i\omega_0\tau_0\theta}.$$

According to

$$[2i\omega_0\tau_0I - \int_{-1}^{0}d\eta(\theta)e^{2i\omega_0\tau_0}]E_1 = f_{z^2},$$

where

$$f_{z^2} = \begin{pmatrix} g_{12}q_1 + g_{13}q_1^2 + g_{14} \\ -2q_3 + g_{22}q_1e^{-i\omega_0\tau_0} + g_{23}e^{-i\omega_0\tau_0} + g_{25}q_1^2e^{-2i\omega_0\tau_0} + g_{26}e^{-2i\omega_0\tau_0} \\ 2uq_3e^{-i\omega_0\tau_0} \end{pmatrix}.$$

We have $E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)})^T$,where $E_1^{(i)} = \dfrac{2\Delta_1^{(i)}}{\Delta_1}(i = 1,2,3)$ with

$$\Delta_1 = 2i\omega_0\tau_0(2i\omega_0\tau_0 - m_{11})(2i\omega_0\tau_0 - n_{22}e^{-2i\omega_0\tau_0}) - 4i\omega_0\tau_0m_{12}n_{21}e^{-2i\omega_0\tau_0}$$
$$- m_{23}n_{32}e^{-2i\omega_0\tau_0}(2ie^{-2i\omega_0\tau_0} - m_{11})$$

$$\Delta_1^{(1)} = (-4\omega_0^2\tau_0^2 - 2in_{22}\omega_0\tau_0)e^{-2i\omega_0\tau_0} - m_{23}n_{32}e^{-2i\omega_0\tau_0}(g_{12}q_1 + g_{13}q_1^2 + g_{14}) - 2m_{12}ie^{-2i\omega_0\tau_0}$$
$$(-2q_3 + g_{22}q_1e^{-2i\omega_0\tau_0} + g_{23}e^{-2i\omega_0\tau_0} + g_{25}q_1^2e^{-2i\omega_0\tau_0} + g_{26}e^{-2i\omega_0\tau_0}) + 2um_{12}m_{23}q_3e^{-2i\omega_0\tau_0},$$

$$\Delta_1^{(2)} = -2in_{21}e^{-2i\omega_0\tau_0}(g_{12}q_1 + g_{13}q_1^2 + g_{14}) + 2i\omega_0\tau_0(2i\omega_0\tau_0 - m_{11})(-2q_3 + g_{22}q_1e^{-i\omega_0\tau_0} + g_{23}e^{-i\omega_0\tau_0} +$$
$$g_{25}q_1^2e^{-2i\omega_0\tau_0} + g_{26}e^{-2i\omega_0\tau_0}) - m_{23}(2i\omega_0\tau_0 - m_{11})2uq_3e^{-2i\omega_0\tau_0},$$

$$\Delta_1^{(3)} = n_{21}n_{32}e^{-4i\omega_0\tau_0}(g_{12}q_1 + g_{13}q_1^2 + g_{14}) - n_{32}e^{-2i\omega_0\tau_0}(2i\omega_0\tau_0 - m_{11})(-2q_3 + g_{22}q_1e^{-i\omega_0\tau_0} + g_{23}e^{-i\omega_0\tau_0} +$$

$$g_{25}q_1^2 e^{-2i\omega_0\tau_0} + g_{26}e^{-2i\omega_0\tau_0}) - (2i\omega_0\tau_0 - m_{11})(2i\omega_0\tau_0 - n_{22}e^{-2i\omega_0\tau_0}$$
$$- m_{12}n_{21}e^{-2i\omega_0\tau_0})2uq_3 e^{-2i\omega_0\tau_0}.$$

And similarly,

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0\tau_0}q(\theta) + \frac{i\overline{g_{11}}}{3\omega_0\tau_0}\bar{q}(\theta) + E_2$$

According to

$$\left(\int_{-1}^{0} d\eta(\theta)\right)E_2 = -f_{z\bar{z}},$$

Where

$$f_{z\bar{z}} = \begin{pmatrix} g_{12}(q_1 + \overline{q_1}) + 2g_{13}q_1\overline{q_1} + 2g_{14} \\ -4\,\mathrm{Re}\{q_3\} + 2g_{22}\,\mathrm{Re}\{q_1 e^{-i\omega_0\tau_0}\} + 2g_{23}\cos(\omega_0\tau_0) + 2g_{25}q_1\overline{q_1} + 2g_{26}\cos(\omega_0\tau_0) \\ 4u\,\mathrm{Re}\{q_3 e^{i\omega_0\tau_0}\} \end{pmatrix},$$

We have $E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)})^T$,where $E_2^{(i)} = \frac{2\Delta_2^{(i)}}{\Delta_2}(i = 1,2,3)$ with

$$\Delta_2 = m_{11}n_{32}m_{23},$$
$$\Delta_2^{(1)} = -n_{32}m_{23}[g_{12}(q_1 + \overline{q_1}) + 2g_{13}q_1\overline{q_1} + 2g_{14}] + 4um_{12}m_{23}\,\mathrm{Re}\{q_3 e^{i\omega_0\tau_0}\},$$
$$\Delta_2^{(2)} = 4m_{11}m_{23}u\,\mathrm{Re}\{q_3 e^{i\omega_0\tau_0}\},$$
$$\Delta_2^{(3)} = n_{21}n_{32}[g_{12}(q_1 + \overline{q_1}) + 2g_{13}q_1\overline{q_1} + 2g_{14}] + m_{11}n_{32}[-4\,\mathrm{Re}\{q_3\} + 2g_{22}\,\mathrm{Re}\{q_1 e^{-i\omega_0\tau_0}\}$$
$$+ 2g_{23}\cos(\omega_0\tau_0) + 2g_{25}q_1\overline{q_1} + 2g_{26}\cos(\omega_0\tau_0)] + 4u(m_{11}n_{22} - m_{12}n_{21})\,\mathrm{Re}\{q_3 e^{i\omega_0\tau_0}\},\ .$$

Consequently, g_{ij} can be expressed explicitly by the parameters and delay in the system (3.1). Thus, we can compute the following values:

$$c_1(0) = \frac{i}{2\omega_0\tau_0}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2},$$

$$\mu_2 = -\frac{\mathrm{Re}(c_1(0))}{\mathrm{Re}(\lambda'(\tau_0))},$$

$$T_2 = -\frac{\mathrm{Im}\,c_1(0) + \mu_2\,\mathrm{Im}\,\lambda'(\tau_0)}{\omega_0\tau_0},$$

$$\beta_2 = 2\,\mathrm{Re}(c_1(0)),$$

which determine the properties of bifurcating periodic solutions at the critical value τ_0. That is, μ_2 determines the direction of Hopf bifurcation: if $\mu_2 > 0$ ($\mu_2 < 0$), then Hopf bifurcation at τ_0 is forward (or backward); β_2 determines the stability of bifurcating periodic solutions: $\beta_2 < 0$ ($\beta_2 > 0$) the bifurcating periodic solution is orbitally asymptotically stable (unstable); and T_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2 > 0$ ($T_2 < 0$).

4. Numerical Simulations and Discussions

In this part, we perform some numerical simulations. We choose a set of parameter which satisfy the assumptions in Theorem 2.1.

$\tau = 2.3767$, $p = 0.362$, $q = 0.199$, $l = 0.1182$, $s = 0.2145$, $u = 0.6209$, $l = 0.1500$.

Thus, all the conditions in Theorem 2.1 are satisfied. Through (2.7), we have $\tau_0 = 5.3774$, $\omega_0 = 0.080$ and $c_1(0) = -0.4617 + i0.6024$. Utilizing Theorem 2.1, we know the equilibrium of the delay model (2.1) is asymptotically stable when $\tau < \tau_0$ (see Figure 1). Hopf bifurcation occurs when $\tau = \tau_0$, and the bifurcating periodic solution is orbitally asymptotically for $\tau > \tau_0$ (see Figure 2).

In addition, the periodic solution of system (2.1) still exists when τ is large. The numerical results show that the global existence of periodic solutions generated by the Hopf bifurcation. How to explain the phenomenon theoretically needs further researches.

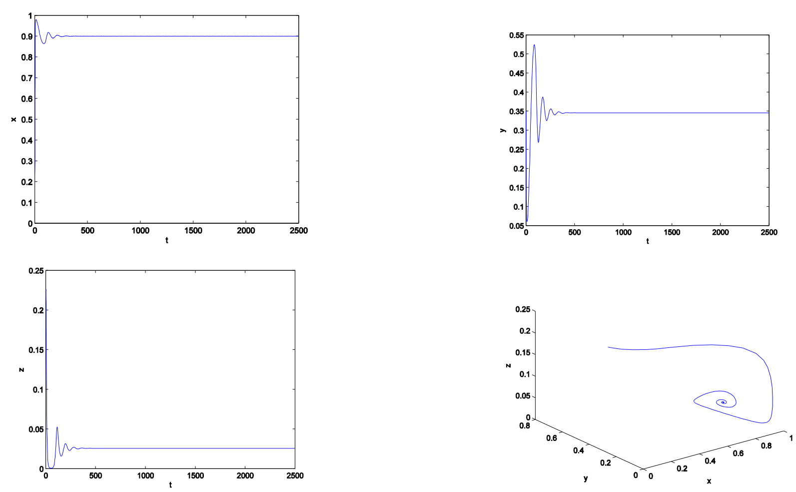
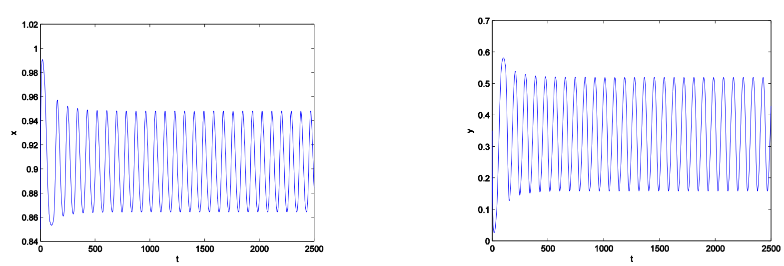


Figure 1. *E*(0.8995, 0.3455, 0.1047) is Asymptotically stable when τ = 2:3767 $< \tau_0$ = 5:3774



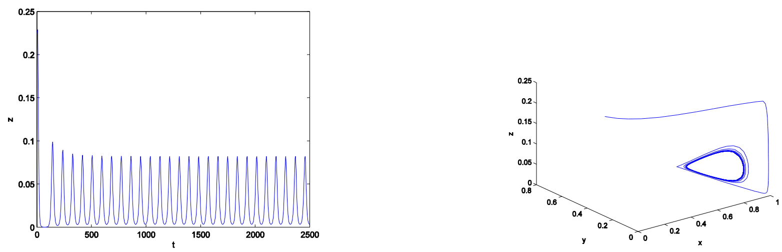


Figure 2. A Stable Periodic Orbit of System (2.1) when τ = 7:8767 > τ_0 = 5:3774

5. Conclusion

In this paper, we analyze the existence of Hopf bifurcation in a class of three-dimensional Gause-type delay ratio-dependent predator-prey model. We obtain the stability of this equilibrium E and also claim that the introduced delay changes its stability while a Hopf bifurcation occurs. The bifurcating periodic solution is asymptotically stable by using the center manifold theorem and the normal form method. The existence of the bifurcation periodic solutions for sufficiently large delay has been shown by numerical simulations. Our investigation shows that the oscillating modes in system (2.1) largely depend on the time delay.

Acknowledgments

The work is supported by the Heilongjiang Provincial Natural Science Foundation (No.A201313).

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