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Reconstruction of a Volatility Based on the Black-Scholes Option Pricing Model Using Homotopy Perturbation Inversion Method

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Abstract

The reconstruction of a volatility based on a Black-Scholes option pricing model is ill-posed. In order to overcome the ill-posedness, a homotopy perturbation inversion method is designed to solve the inverse problem. The proposed method is a modified version of the Landweber method. The reconstruction of a volatility is a nonlinear problem which is needed to be linearized. Hence, numerical experiments consist of the reconstruction of a policy parameter based on a Todaro model which is a linear inverse problem and the reconstruction of a volatility based on a Black-Scholes option pricing model in order to test the performance of the proposed method. Numerical examples show that the proposed method is more accurate and faster than the Landweber method.

***Keywords**: Option Pricing, Black-Scholes Model, Inverse Problem, Volatility, Homotopy Perturbation Inversion Method*

1. Introduction

Inverse problems play the important role in the financial mathematics, such as the reconstruction of a volatility. With the development of economy and financial mathematics, the reconstruction of a volatility for the option pricing has been widely used in many real applications. Option is a popular type of financial derivatives, and provides us opportunities for buying or selling a certain number of basic commodity options in the future.

In the option trading market, the option price is determined from option trading sellers and purchasers in the way of a competitive bidding based on the applications of computers and advanced communication technologies. In the theoretical field, Fischer Black and Myron Scholes provided the first complete option pricing model (the so-called Black-Scholes (B-S) formula) [1]. Following the framework of the B-S formula, the inverse problem of an option pricing is to reconstruct the risk neutral measure by using the measurement data [2].

In the option trading market, the performance of an implied volatility often shows the two kinds of curves: a "smile" curve and a "skew" curve which correspond to the exercise price and to the maturity, respectively. The reconstruction of a financial parameter is ill-posed, in other words, a very small noise of the measurement data may lead to a very large error of the reconstruction. For different maturities and strike prices, the volatility is more stable than options or stock prices. Therefore, the reconstruction of a volatility plays an important role in the financial market.

Volatility reconstruction is the important part of the B-S Theory. In 2005, Hein provided the analysis of Tikhonov regularization method for the inverse problem of option pricing in the price-dependent case [3]. Egger and Engl gave the convergence

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analysis and stability of the inverse problem of option pricing by using a Tikhonov regularization method [4]. In 2014, Trong *et al*. investigated an explicit formula of the implied volatility [5]. In real applications, a Landweber method as an iterative regularization method is used to solve large scale problems. The stability of a Landweber method is superior to other methods, however, it has a slow convergent rate and a low accuracy. In order to improve the accuracy and convergence rate of a Landweber method, a homotopy perturbation inversion method is designed to reconstruct a volatility.

This paper discusses the reconstruction of a volatility based on the Black-Scholes option pricing model. A finite difference method is used to solve the forward problem. The reconstruction of a volatility is a nonlinear inverse problem, and hence the reconstruction of a policy parameter based on a Todaro model (which is a linear inverse problem) is investigated in numerical experiments in order to test the performance of the proposed method for linear and nonlinear cases.

2. Mathematical Model

The B-S formula is widely used in the field of a derivative pricing, when the price changes of a derivative satisfy the standard geometric Brown motion. The boundary conditions of the different values vary with the different types of derivatives. When boundary conditions are given, a derivative pricing V_t can be obtained by solving the B-S formula of the derivative pricing model. When the derivative is taken as option, the forward problem is defined as the determination of the option pricing. For simplicity, the European call option is taken as example.

The relationship between the European call and put options shows that the reconstructed volatility should be the same by using the call options market quotes or the put options market quotes. For the European call option in the time interval $[0,T]$, let $V_t = V(S_t, t; \sigma, K, T)$ be the European call option pricing in the domain $\sum : \{0 \le S < \infty, 0 \le t \le T\}$, and V_t satisfies [6,7]

$$\begin{cases} \dfrac{\partial V_t}{\partial t} + (r-q)S_t \dfrac{\partial V_t}{\partial S_t} + \dfrac{1}{2}\sigma^2(S_t,t)S_t^2 \dfrac{\partial^2 V_t}{\partial S_t^2} - rV_t = 0 \\ V_T = (S_t - K)^+. \end{cases} \tag{1}$$

where S : stock prices, K : strike price, r : interest rate, q : dividend, T : availability period, t : time, σ : volatility.

In the B-S formula, except the volatility is a free variable, the other parameters and variables are decided by the prevailing market conditions or contracts. Hence, the volatility σ is a very important parameter. For an option, the volatility can be reconstructed from the B-S formula. The inverse problem is defined as follows: reconstructing the volatility $\sigma = \sigma(S_t, t)$ from the measurement data $V(S_*, t_*; \sigma, K_i, T_i) = V_i$, $(i = 1, \cdots, N)$, $t = t_* (0 \le t_* \le T)$, $S = S_*$.

3. Solving Forward Problem using a Finite Difference Method

A finite difference method is used to solve the B-S formula. Many methods solving the forward problem of the B-S formula exist in many references; however, the finite difference method is briefly provided for convenience.

Let $\tilde{S} = \{0, \delta S, 2\delta S, 3\delta S, \cdots, M\delta S\}$, $\tilde{T} = \{0, \delta t, 2\delta t, 3\delta t, \cdots, M\delta t\}$,

where $M\delta S = S_{\max}$, $N\delta t = T$.

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The boundary conditions of the European call put option are taken as

$$V(S,T)=\max(K-S,0)$$
$$V(0,t)=Ke^{-r(T-t)}$$
$$V(S_{\max},t)=0. \tag{2}$$

Let

$$V_{i,N}=\max(K-(i\delta S),0)\ , \qquad i=0,1,\cdots,M,$$
$$V_{0,j}=Ke^{-r(N-j)\delta t}, \qquad j=0,1,\cdots,N,$$
$$V_{M,j}=0, \qquad j=0,1,\cdots,N. \tag{3}$$

The B-S formula is in the discrete form

$$\frac{V_{i,j}-V_{i,j-1}}{\delta t}+\frac{1}{2}\sigma^2(i\delta S)^2\frac{V_{i+1,j-1}-2V_{i,j-1}+V_{i-1,j-1}}{\delta S^2}$$
$$+r(i\delta S)\frac{V_{i+1,j-1}-V_{i-1,j-1}}{2\delta S}-rV_{i,j-1}=0. \tag{4}$$

Equation (4) is written as

$$V_{i,j}=A_iV_{i-1,j-1}+B_iV_{i,j-1}+C_iV_{i+1,j-1}, \tag{5}$$

where

$$A_i=\frac{1}{2}\delta t(ri-\sigma^2i^2),$$
$$B_i=1+(\sigma^2i^2+r)\delta t,$$
$$C_i=-\frac{1}{2}\delta t(ri+\sigma^2i^2). \tag{6}$$

For the forward problem, Eqs. (4) and (5) are used to obtain $V_{i,j}$ when σ is known.

4. A Landweber Method for the Inverse Problem

A nonlinear vector-valued function is defined as $F:\sigma\rightarrow V_0$, namely, $F(\sigma)=V_0$. Let

$$\sigma^T=\left(\sigma(0,0),\sigma(0,1),\cdots,\sigma(0,m),\cdots,\sigma(n,0),\sigma(n,1),\cdots,\sigma(n,m)\right),$$
$$V_0^T=\left(V_0(0,0),V_0(0,1),\cdots,V_0(0,m),\cdots,V_0(n,0),V_0(n,1),\cdots,V_0(n,m)\right).$$

Because the inverse problem is ill-posed, F is a nonlinear ill-posed operator. To reconstruct the volatility, the cost function is as follows [8-10]

$$J(\alpha)=\left\|F(\sigma)-V_0\right\|_2^2. \tag{7}$$

The method to solve the cost function plays an important role in inverse problems. A popular algorithm-a Landweber method is briefly provided.

The Landweber method arises from the gradient-like numerical method. This method does not invert the F-derivative, and hence it is widely used in large scale problems.

The Landweber method can be written as

$$\sigma_{n+1}^{\delta}=\sigma_n^{\delta}-F^{'}(\sigma_n^{\delta})^*(F(\sigma_n^{\delta})-V_0). \tag{8}$$

For large scale problems, $F^{'}(\sigma_n^{\delta})^*(F(\sigma_n^{\delta})-V_0)$ can be determined by using the adjoint method. Although the Landweber method is easy to implement and is stable to noises, it is very slow for many problems.

In order to guarantee the convergence, the following modified version is used to solve the real applications:

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$$\sigma_{n+1}^{\delta} = \sigma_n^{\delta} - \omega F^{'}(\sigma_n^{\delta})^{*}(F(\sigma_n^{\delta}) - V_0), \tag{9}$$

where ω is a damped parameter satisfying
$\left\|\sigma_{n+1}^{\delta} - \sigma_n^{\delta}\right\| < \left\|\sigma_n^{\delta} - \sigma_{n-1}^{\delta}\right\|$.

5. A Homotopy Perturbation Inversion Method based on a Landweber Method

The nonlinear ill-posed operator equation $F(\sigma) = V_0$ is transformed to the Euler equation

$$F^{'}(\sigma)^{*}(F(\sigma) - V_0) = 0, \tag{10}$$

where $F^{'}(\sigma)$ is the F-derivative.

Setting a homotopy mapping $H : F \times [0,1] \to Y$

$$H(x, p) = p[F^{'}(x)^{*}(F(x) - V_0)] + (1-p)(x - \sigma_0) = 0,\ p \in [0,1] \tag{11}$$

where p : embedding parameter, V_0 : an initial guess value. Hence,

$$\begin{aligned} &H(x,0) = x - \sigma_0 = 0, \\ &H(x,1) = F^{'}(x)^{*}(F(x) - V_0) = 0. \end{aligned} \tag{12}$$

σ is taken as the power series of p :

$$x = \sigma_0 + p\sigma_1 + p^2\sigma_2 + \cdots, \tag{13}$$

and the approximation of Equation (7) is obtained as

$$\sigma = \lim_{p \to 1} x = \sigma_0 + \sigma_1 + \sigma_2 + \cdots \tag{14}$$

The operator $F(x)$ in Equation (11) is expanded as a Taylor series near σ_0 :

$$\begin{aligned} H(x, p) = {} & p[F^{'}(\sigma_0)^{*}(F(\sigma_0) + F^{'}(\sigma_0)(x - \sigma_0) \\ & + o(x - \sigma_0)^2 - V_0)] + (1-p)(x - \sigma_0) = 0, \end{aligned} \tag{15}$$

$$\begin{aligned} & p[F^{'}(\sigma_0)^{*}(F(\sigma_0) + F^{'}(\sigma_0)(p\sigma_1 + p^2\sigma_2 + p^3\sigma_3 + \cdots) - V_0) \\ & + \alpha(\sigma_0 + p\sigma_1 + p^2\sigma_2 + p^3\sigma_3 + \cdots)] \\ & + (1-p)(p\sigma_1 + p^2\sigma_2 + p^3\sigma_3 + \cdots) = 0, \end{aligned}$$

Following the power of p , one can get

$$\begin{aligned} & p^1 : \sigma_1 = -F^{'}(\sigma_0)^{*}(F(\sigma_0) - V_0) \\ & p^2 : \sigma_2 = (I - F^{'}(\sigma_0)^{*} F^{'}(\sigma_0))(-F^{'}(\sigma_0)^{*}(F(\sigma_0) - V_0)) \end{aligned} \tag{16}$$

and

$$\begin{aligned} \sigma &= \lim_{p \to 1} v = \sigma_0 + \sigma_1 + \cdots \\ &= \sigma_0 - F^{'}(\sigma_0)^{*}(F(\sigma_0) - V_0) \\ &\quad + (I - F^{'}(\sigma_0)^{*} F^{'}(\sigma_0))(-F^{'}(\sigma_0)^{*}(F(\sigma_0) - V_0)) + \cdots. \end{aligned} \tag{17}$$

The volatility σ for the noisy measurement data is reconstructed by the first two terms:

$$\sigma_{n+1}^{\delta} = \sigma_n^{\delta} - (2I - F^{'}(\sigma_n^{\delta})^{*} F^{'}(\sigma_n^{\delta}))F^{'}(\sigma_n^{\delta})^{*}(F(\sigma_n^{\delta}) - V_0), \tag{18}$$

where $\left\|V_0^{\delta} - V_0\right\| \le \delta$ and the parameter n denotes the iteration number.

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When the volatility σ is reconstructed by the first term:

$$\sigma_{n+1}^{\delta}=\sigma_{n}^{\delta}-F^{'}(\sigma_{n}^{\delta})^{*}(F(\sigma_{n}^{\delta})-V_{0}), \tag{19}$$

Equation (19) is a well-known Landweber method that is stable; however, the convergence rate of this method is very slow. Equation (18) is a modified version of Equation (19), which is called a homotopy perturbation inversion method (HPIM). Because HPIM is a modified version, the convergence rate of HPIM is faster than a standard Landweber method.

6. Extension to HPIM based on a Gauss-Newton Method

The formula (18) is designed from the Landweber method. Another well-known method is called as a Gauss-Newton method, which is based on the regularized cost function

$$J(\alpha)=\|F(\sigma)-V_{0}\|_{2}^{2}+\alpha\|\sigma\|_{2}^{2}, \tag{20}$$

where $\|F(\sigma)-V_{0}\|$ is a misfit term , $\|\sigma\|$ is a penalty term, and α is a regularized parameter, which balances the misfit term and the penalty term. The Gauss-Newton method solves the above cost function as follows:

$$\sigma_{n+1}^{\delta}=\sigma_{n}^{\delta}-[F^{'}(\sigma_{n}^{\delta})^{*}F^{'}(\sigma_{n}^{\delta})+\alpha_{k}I]^{-1}[F^{'}(\sigma_{n}^{\delta})^{*}(F(\sigma_{n}^{\delta})-V_{0})+\alpha_{k}\sigma_{n}^{\delta}]. \tag{21}$$

Like the Landweber method, the following modified version is used to solve the real applications:

$$\sigma_{n+1}^{\delta}=\sigma_{n}^{\delta}-\omega[F^{'}(\sigma_{n}^{\delta})^{*}F^{'}(\sigma_{n}^{\delta})+\alpha_{k}I]^{-1}[F^{'}(\sigma_{n}^{\delta})^{*}(F(\sigma_{n}^{\delta})-V_{0})+\alpha_{k}\sigma_{n}^{\delta}], \tag{22}$$

where ω is a damped parameter satisfying

$\|\sigma_{n+1}^{\delta}-\sigma_{n}^{\delta}\|<\|\sigma_{n}^{\delta}-\sigma_{n-1}^{\delta}\|.$

The accuracy and convergence rate of the Gauss-Newton method are better than those of the Landweber method, however, the stability of the Gauss-Newton method is worse than that of the Landweber method. Because it is very difficult to implement the $F^{'}(\sigma)$ in large scale problems, the Gauss-Newton method is widely used in small and medium scale problems.

The nonlinear ill-posed operator equation $F(\sigma)=V_{0}$ is transformed to the Euler equation

$$F^{'}(\sigma)^{*}(F(\sigma)-V_{0})+\alpha\sigma=0, \tag{23}$$

where $F^{'}(\sigma)$ is the F-derivative.

Setting a homotopy mapping $H:F\times[0,1]\to Y$

$$H(x,p)=p[F^{'}(x)^{*}(F(x)-V_{0})+\alpha x]+(1-p)(x-\sigma_{0})=0,\ p\in[0,1] \tag{24}$$

where p : embedding parameter, V_{0} : an initial guess value. Hence,

$$\begin{aligned}H(x,0)&=x-\sigma_{0}=0,\\ H(x,1)&=F^{'}(x)^{*}(F(x)-V_{0})+\alpha x=0.\end{aligned} \tag{25}$$

σ is taken as the power series of p :

$$x=\sigma_{0}+p\sigma_{1}+p^{2}\sigma_{2}+\cdots, \tag{26}$$

and the approximation of Equation (20) is obtained as

$$\sigma=\lim_{p\to 1}x=\sigma_{0}+\sigma_{1}+\sigma_{2}+\cdots \tag{27}$$

The operator $F(x)$ in Equation (24) is expanded as a Taylor series near σ_{0} :

$$H(x,p)=p[F^{'}(\sigma_0)^{*}(F(\sigma_0)+F^{'}(\sigma_0)(x-\sigma_0)$$
$$+o(x-\sigma_0)^2-V_0)+\alpha x]+(1-p)(x-\sigma_0)=0,$$

$$\begin{aligned}&p[F^{'}(\sigma_0)^{*}(F(\sigma_0)+F^{'}(\sigma_0)(p\sigma_1+p^2\sigma_2+p^3\sigma_3+\cdots)-V_0)\\&\quad+\alpha(\sigma_0+p\sigma_1+p^2\sigma_2+p^3\sigma_3+\cdots)]\\&\quad+(1-p)(p\sigma_1+p^2\sigma_2+p^3\sigma_3+\cdots)=0,\end{aligned} \tag{28}$$

Following the power of p , one can get

$$\begin{aligned}&p^1:\sigma_1=-F^{'}(\sigma_0)^{*}(F(\sigma_0)-V_0)-\alpha\sigma_0\\&p^2:\sigma_2=(I-F^{'}(\sigma_0)^{*}F^{'}(\sigma_0))(-F^{'}(\sigma_0)^{*}(F(\sigma_0)-V_0))\\&\qquad-\alpha(F^{'}(\sigma_0)^{*}(F(\sigma_0)-V_0)-\alpha\sigma_0)\end{aligned} \tag{29}$$

and

$$\begin{aligned}\sigma&=\lim_{p\to 1}v=\sigma_0+\sigma_1+\cdots\\&=\sigma_0-F^{'}(\sigma_0)^{*}(F(\sigma_0)-V_0)-\alpha\sigma_0\\&+(I-F^{'}(\sigma_0)^{*}F^{'}(\sigma_0))(-F^{'}(\sigma_0)^{*}(F(\sigma_0)-V_0))\\&+\ -\alpha(F^{'}(\sigma_0)^{*}(F(\sigma_0)-V_0)-\alpha\sigma_0)\ +\cdots.\end{aligned} \tag{30}$$

7. Numerical Experiments

In this section, some numerical experiments are run to test the accuracy and efficiency of the proposed method compared with a standard Landweber method, as the same as with a Gauss-Newton method. To guarantee the convergence of HPIM, the formula (18) is modified as

$$\sigma_{n+1}^{\delta}=\sigma_n^{\delta}-[2I-F^{'}(\sigma_n^{\delta})^{*}F^{'}(\sigma_n^{\delta})/\lambda_{\max}(F^{'}(\sigma_n^{\delta})^{*}F^{'}(\sigma_n^{\delta}))]F^{'}(\sigma_n^{\delta})^{*}(F(\sigma_n^{\delta})-V_0), \tag{31}$$

where $\lambda_{\max}(F^{'}(\sigma_n^{\delta})^{*}F^{'}(\sigma_n^{\delta}))$ denotes the maximum eigenvalue. The stopping criteria are taken as

$$\begin{aligned}&\|F(\sigma_n)-V_0\|\le\tau\delta,\ \tau=1.05,\\&\frac{\|\sigma_{n+1}-\sigma_n\|}{\|\sigma_n\|}\le 10^{-6}.\end{aligned} \tag{32}$$

7.1. Reconstruction of policy Parameter based on a Todaro Model (a linear case)

Todaro model is the famous model to describe the number of rural migrants (namely, workers in urban areas from rural areas) in the labor economics. A Todaro model shows the relationship between the number of rural migrants with the income difference of rural and urban areas. A Todaro model can be written in the following form

$$M=f(d), \tag{33}$$

where M,d denote the number of rural migrants and the income difference of rural and urban areas, respectively. The function f is an increasing function, *i.e.,* $f^{'}>0$. A modified Todaro model considers policy parameter χ

$$M=f(\chi,d), \tag{34}$$

where χ describes the efficiency of government policy.

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I rural areas and J urban areas are considered. d_{ij} ($1 \le i \le I, 1 \le j \le J$) stands for the income difference between the i th rural area and the j th urban area, and $g_{j\tilde{j}}$ ($1 \le j \le J, 1 \le \tilde{j} \le J$) stands for the income difference between the j th and the $\tilde{j}$ th urban areas. The number of migrants into the j th urban area denotes M_j. For the j th urban area, policy parameter χ_j is split into two parts $\chi_j = \chi_j^r + \chi_j^u$, where χ_j^r, χ_j^u denote the efficiency of government policy with respect to rural and urban areas, respectively. The number of workers from urban areas into rural areas is set to zero. In real applications, the function f has many different representations. In this section, we focus on the performance of the proposed method, and hence f is taken as a linear function. The Todaro model is modified as

$$\begin{array}{c}
\chi_1^r d_{11} + \chi_1^r d_{21} + \cdots + \chi_1^r d_{i1} + \cdots + \chi_1^r d_{I1} + \chi_1^u g_{11} + \chi_1^u g_{21} + \cdots + \chi_1^u g_{j1} + \cdots + \chi_1^u g_{J1} = M_1 \\
\cdots \qquad \cdots \qquad \cdots \\
\chi_j^r d_{1j} + \chi_j^r d_{2j} + \cdots + \chi_j^r d_{ij} + \cdots + \chi_j^r d_{Ij} + \chi_j^u g_{1j} + \chi_j^u g_{2j} + \cdots + \chi_j^u g_{jj} + \cdots + \chi_j^u g_{Jj} = M_j \\
\cdots \qquad \cdots \qquad \cdots \\
\chi_J^r d_{1J} + \chi_J^r d_{2J} + \cdots + \chi_J^r d_{iJ} + \cdots + \chi_J^r d_{IJ} + \chi_J^u g_{1J} + \chi_J^u g_{2J} + \cdots + \chi_J^u g_{jJ} + \cdots + \chi_J^u g_{JJ} = M_J
\end{array} \tag{35}$$

where the income differences d_{ij} and $g_{j\tilde{j}}$, and the number of migrants into the j th urban area M_j are known. The policy parameter $X = (\chi_1^r, \chi_1^u, \chi_2^r, \chi_2^u, \cdots, \chi_j^r, \chi_j^u, \cdots, \chi_J^r, \chi_J^u)^T$ is unknown.

Let $D_j = \sum_{i=1}^{I} d_{ij}$, $G_j = \sum_{j=1}^{J} g_{j\tilde{j}}$, and

$$A = \begin{pmatrix} D_1 & G_1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & D_j & G_j & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & D_J & G_J \end{pmatrix} \tag{36}$$

$$M = \left(M_1, M_2, \cdots M_j, \cdots M_J\right)^T$$

Equation (35) is recast as

$$AX = M \tag{37}$$

In numerical tests, we take $I = 30, J = 10$ and add 0%, 0.5%, 1% Gaussian random noises to the measurement data in order to test the stability. The relative errors are shown in Table 1. Table 1 shows that the performance of the proposed method than that of the Landweber method.

Table 1. Relative Errors

Noise	HPIM	Landweber
0%	0.58%	0.98%
0.5%	1.66%	2.87%
1%	3.09%	5.81%

7.2. Reconstruction of Volatility Based on A Black-Scholes Option Pricing Model (a Nonlinear Case)

For testing the performances of HPIM and Landweber method, we set $T = 1, 2, 3$, respectively, and set stock prices $S = 50$, interest rate $r = 0.05$. We investigate the effect of different measurement data on reconstruction. A 1% Gaussian random noise is added to the measurement data in order to test the stability. In Table 2, the exact and initial guess volatilities are provided for $T = 1, 2, 3$. Let $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_L)$ be a piece-wise constant volatility with the equally-spaced time interval. Three exact volatilities are chosen in numerical experiments, respectively,

Case 1:

$$\sigma = 0.5000, \quad t \in (0, T);$$

Case 2:

$$\sigma = \begin{cases} 0.2000, & t \in (0,\ T/2); \\ 0.4000, & t \in (T/2,\ T); \end{cases}$$

Case 3:

$$\sigma = \begin{cases} 0.2500, & t \in (0,\ T/3); \\ 0.2000, & t \in (T/3,\ 2T/3); \\ 0.3000, & t \in (2T/3,\ T). \end{cases}$$

Due to the nonlinearity and ill-posedness, the different initial guess volatilities are chosen for reconstruction in different cases:

Case 1:

$$\sigma = 0.1000, \quad t \in (0, T);$$

Case 2:

$$\sigma = 0.4500, \quad t \in (0, T);$$

Case 3:

$$\sigma = 0.3000, \quad t \in (0, T).$$

For $T = 1$, the reconstructed results by using the Landweber method and HPIM are given in Table 3. Table 4 shows the computational times of the Landweber method and HPIM. Reconstruction results for $T = 2, 3$ show in Tables 5-6 and Tables 7-8, respectively. For Case 1 and Case 2, the accuracy of HPIM is better than one of the Landweber method, because HPIM is a modified version of the Landweber method. The computational time of HPIM is less than that of the Landweber method. The computational times of them are the same for the exact volatility $(0.2500, 0.2000, 0.3000)$; however, the reconstruction of HPIM is much closer to the exact value.

Table 2. Exact and Initial Guess Volatilities ($T = 1, 2, 3$)

K	Exact volatility	Initial guess volatility
50	0.5000	0.1000
50,45	(0.2000,0.4000)	(0.4500,0.4500)
50,45,40	(0.2500, 0.2000,0.3000)	(0.3000, 0.3000,0.3000)

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Table 3. Reconstruction Results (Continued $T=1$)

Reconstructed volatility(Landweber)	Reconstructed volatility(HPIM)
0.5003	0.5000
(0.2000,0.3997)	(0.2000,0.3999)
(0.2580, 0.2000,0.2999)	(0.2517, 0.2000,0.3000)

Table 4. Computational Times (continued $T=1$)

Time(s) (Landweber)	Time(s) (HPIM)
2.7653	0.9380
6.3128	5.1286
4.9863	5.090

Table 5. Reconstruction Results (Continued $T=2$)

Reconstructed volatility(Landweber)	Reconstructed volatility(HPIM)
0.5005	0.5002
(0.2001,0.3995)	(0.2000,0.3997)
(0.2619, 0.2000,0.2996)	(0.2572, 0.2000,0.2999)

Table 6. Computational Times (Continued $T=2$)

Time(s) (Landweber)	Time(s) (HPIM)
3.5370	1.2742
8.8112	6.5231
5.6530	5.8033

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Table 7. Reconstruction Results (Continued $T = 3$**)**

Reconstructed volatility(Landweber)	Reconstructed volatility(HPIM)
0.5012	0.5006
(0.1998,0.3993)	(0.2000,0.3996)
(0.2774, 0.2000,0.2994)	(0.2581, 0.2000,0.2997)

Table 8. Times (Continued $T = 3$**)**

Time(s) (Landweber)	Time(s) (HPIM)
4.7452	1.5208
10.6663	7.8146
6.2702	6.3508

8. Conclusions

Both the reconstruction of a volatility based on the Black-Scholes option pricing model and the reconstruction of a policy parameter based on the Todaro model are investigated. Since the inverse problem is ill-posed, a regularization method is used to overcome the ill-posedness. The well-known Landweber method is very slow, and hence a new method is designed to speed up the Landweber method. Numerical experiments show that the proposed method HPIM is faster and more accuracy than the Landweber method.

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