International Journal of Hybrid Information Technology
Vol. 9, No.8 (2016), pp. 1-10
http://dx.doi.org/10.14257/ijhit.2016.9.8.01

A New Method for Reducing Concept Lattice Attributes Using Infimum Irreducible Concepts

Wu Jie[1], Liang Yan and Ma Yuan[2]

[1,2]*School of Software, University of Science and Technology Liaoning, Liaoning, China*
School of Applied Technology, University of Science and Technology Liaoning, Liaoning, China
wujieaa@163.com

Abstract

Attribute reduction have always been hot issues in the concept lattice research. In this paper, we define the attribute waned value, and prove that using attribute waned values can simplifies the conversion of the discernibility function. Furthermore, we prove the attribute reduction can be generated by selecting elements from attribute waned values of infimum irreducible concepts, and provide an approach that uses attribute concepts to find all infimum irreducible concepts. In addition, we give an algorithm to get all attribute reductions. Because this algorithm only uses infimum irreducible concepts rather than all concepts, the time and space complexity is polynomial form and smaller.

***Keywords**: Concept lattice; Attribute waned value; Infimum irreducible concept; Attribute reduction*

1. Introduction

Concept lattice belongs to the mathematics concept and the hierarchy of concepts in the field of Applied Mathematics[1]. Since concept lattice is strictly hierarchical and it can easily describe generalization and specialization among things, it has many successful applications, such as the spatial clustering [2], symptom intelligent diagnosis, Folksonomy, information revi-sion and file Browser, software evolution analysis, access management, proposition reduction and so on. Although concept lattice has been broadly applied in many areas, the problem is that the number of concept in formal context is the exponential growth in the wake of the size of the context (For instance, $(S,\leq)$ is a partially ordered set, concept number of reverse rated ruler $\mathbb{N}_s^c = (S,S,\not\geq)$ is $2^{|S|}$).[3] If the context is slightly bigger, the concept is hard to be calculated and the problem solution becomes difficult. Therefore, it is important to make the formal context to be irreducible. In recent years, there are mainly following works about the concept lattice reduction: the reduction to make sure the concept lattice is an isomorphism[4,5], the reduction to make sure the equivalence classes of objects are constant[6], the reduction to make sure the extents of objects are constant[7-9], and the reduction to make sure the decision rules are complete[10]. Although above reduction methods have unique design and have made a great success in some applications, to find all the possible reduction, the calculation time of the methods is exponential. At present, there are a lot of new ideas and methods for the reduction of the concept context, such as the reduction method based on axialities [11], approximate concepts acquisition method based on k-grade relation object set [12], homomorphism reduction of consistent decision context[13], the object-oriented reduction method based on attribute rank of concept[14], the method based on covering of the object (attribute) set [15].

ISSN: 1738-9968 IJHIT

Reference [4] provides an important method to find out all attribute reductions for a given formal context. But the algorithm complexity is exponential in [4]. With the increase of formal contexts, it is very difficult to find out all attribute reductions. Meanwhile, the method is hard to understand, since it cannot be shown in Hasse graphs. In this paper, we present a reduction method based on infimum irreducible concepts. The operation process of this method is simpler, the display in the Hasse diagram is intuitive and easy, and the form of time complexity is polynomial. The new approach improves the algorithm complexity in [4].

The rest of the paper is organized as follows. Section 2 introduces the main definition and theorem of concept lattice. Section 3 defines the waned value. Section 4 writes about the infimum irreducible and the related theorems. Section 5 shows a new attribute reduction algorithm and analyzes the complexity of the algorithm. Section 6 concludes the paper with a summary.

2. Basic Definitions and Theorems of Concept Lattice

Definition 1[16]**.** A formal context $\mathrm{K}=(U,M,I)$ consists of two sets G, M and a relation $I\subseteq U\times M$. The elements of G are called the objects and the elements of M are called the attributes of the context.

Definition 2[16]**.** Let $\mathrm{K}=(U,M,I)$ be a formal context, $A\subseteq U$, $B\subseteq M$,

$$f(A)=\left\{m\in M \,\middle|\, \forall u\in A,(u,m)\in I\right\}$$

$$g(B)=\left\{u\in U \,\middle|\, \forall m\in B,(u,m)\in I\right\}$$

The tuple (A,B) is called a concept, if and only if $f(A)=B$, $g(B)=A$. A and B are called as extension and intention respectively. The set of all concepts of K is denoted by $\mathfrak{B}(\mathrm{K})$.

Property 1[16]**.** Let $\mathrm{K}=(U,M,I)$ be a formal context, $A_1,A_2\subseteq U$, $B_1,B_2\subseteq M$, then

$$A_1\subseteq A_2\Rightarrow f(A_2)\subseteq f(A_1) \quad (1)$$

$$A_1\subseteq A_2\Rightarrow g(A_2)\subseteq g(A_1) \quad (2)$$

$$A_1\subseteq g(f(A_1)) \quad (3)$$

$$B_1\subseteq f(g(B_1)) \quad (4)$$

$$f(A_1)=f(g(f(A_1))) \quad (5)$$

$$g(B_1)=g(f(g(B_1))) \quad (6)$$

$$f(A_1)\cap f(A_2)=f(A_1\cup A_2) \quad (7)$$

$$g(B_1)\cap g(B_2)=g(B_1\cup B_2) \quad (8)$$

By the property 1(5), $\forall A\subseteq U$, $(g(f(A)),f(A))$ must be a concept. By the property 1(6), $\forall B\subseteq M$, $(g(B),f(g(B)))$ must be a concept. Specially, if A has only one object u, then $(g(f(u)),f(u))$ is called the object concept of u. If B has only one attribute m, then $(g(m),f(g(m)))$ is called the attribute concept of m.

Definition 3[16]. Let $\mathrm{K}=(U,M,I)$ be a formal context. (X_1,Y_1), $(X_2,Y_2)\in\mathfrak{B}(\mathrm{K})$. If $X_1\subseteq X_2$, (X_1,Y_1) is called the subconcept of (X_2,Y_2), and (X_2,Y_2) is the superconcept of (X_1,Y_1), the relation between two concepts is denoted as $(X_1,Y_1)\le(X_2,Y_2)$. If $X_1\subset X_2$, the relation is denoted as $(X_1,Y_1)<(X_2,Y_2)$. If $(X_1,Y_1)<(X_2,Y_2)$ and there is no (X_3,Y_3) with $(X_1,Y_1)<(X_3,Y_3)<(X_2,Y_2)$, then

(X_1,Y_1) is the direct subconcept of (X_2,Y_2), (X_2,Y_2) is the direct superconcept of (X_1,Y_1), and the relation of two concepts is denoted as $(X_1,Y_1) \prec (X_2,Y_2)$.

Definition 4[16]**.** Let be a formal context. $\mathcal{D} \subseteq \mathfrak{B}(\mathrm{K})$, $\exists (X_0,Y_0) \in \mathfrak{B}(\mathrm{K})$, $\forall (X,Y) \in \mathcal{D}$, $(X,Y) \geq (X_0,Y_0)$, (X_0,Y_0) is called a lower bound of $\mathcal{D}$. If the set of all lower bounds of $\mathcal{D}$ has a maximum element, the maximum element is called the infimum of $\mathcal{D}$, denoted as $\wedge \mathcal{D}$. If there are only two element (X_1,Y_1) and (X_2,Y_2) in $\mathcal{D}$, the infimum is denoted as $(X_1,Y_1) \wedge (X_2,Y_2)$. $\exists (X_0,Y_0) \in \mathfrak{B}(\mathrm{K})$, $\forall (X,Y) \in \mathcal{D}$, $(X,Y) \leq (X_0,Y_0)$, (X_0,Y_0) is an upper bound of $\mathcal{D}$. If the set of all upper bounds of $\mathcal{D}$ has a minimum element, the minimum element is called as the supremum of $\mathcal{D}$, denoted as $\vee \mathcal{D}$. If there are only two element (X_1,Y_1) and (X_2,Y_2) in $\mathcal{D}$, the supremum is denoted as $(X_1,Y_1) \wedge (X_2,Y_2)$.

Definition 5[16]**.** Let $\mathrm{K} = (U,M,I)$ be a formal context. If $(X_0,Y_0) \in \mathfrak{B}(\mathrm{K})$ and $(X_0,Y_0) \neq \wedge \{(X,Y) \in \mathfrak{B}(\mathrm{K}) \mid (X,Y) > (X_0,Y_0)\}$, then (X_0,Y_0) is an Infimum Irreducible concept.

Example 1. Let $\mathrm{K} = (U,M,I)$ be a given formal context (see Table 1). The Hasse graph of the context is shown in Figure 1. The concept #7 $(1268, adk)$ is an infimum irreducible concept because #7 $(1268, adk) \neq \wedge \{(X,Y) \in \mathfrak{B}(\mathrm{K}) \mid (X,Y) > (1268, adk)\}$ = #2 $(123689, a)$, while the concept #5 $(138, abe)$ is not an infimum irreducible concept because $\wedge \{(X,Y) \in \mathfrak{B}(\mathrm{K}) \mid (X,Y) > (138, abe)\} = \wedge \{\#2(123689, a), \#3(134578, b)\} = \#5\ (138, abe)$.

Lemma 1[16]**.** Let $\mathrm{K} = (U,M,I)$ be a formal context. Both (X_1,Y_1) and (X_2,Y_2) are concepts. Then the supremum of (X_1,Y_1) and (X_2,Y_2) is $(\mathrm{g}(Y_1 \cap Y_2), Y_1 \cap Y_2)$, and the infimum is $(X_1 \cap X_2, f(X_1 \cap X_2))$.

Definition 6[4]. Let $\mathrm{K} = (U,M,I)$ be a formal context, (A_i, B_i), $(A_j, B_j) \in \mathfrak{B}(\mathrm{K})$. discernibility attribute set is $DIS((A_i,B_i),(A_j,B_j)) = B_i \cup B_j - B_i \cap B_j$. Discernibility matrix is $\Lambda_{FC} = (DIS((A_i,B_i),(A_j,B_j)),(A_i,B_i),(A_j,B_j) \in L(U,M,I))$. discernibility function is

$$F(\Lambda_{FC}) = \underset{H \in \Lambda_{FC}}{\wedge} (\underset{h \in H}{\vee} h).$$

Table 1. A Given Formal Context

	a	b	c	d	e	f	g	h	i	j	k	l
1	×	×		×	×	×	×		×	×	×	×
2	×			×							×	
3	×	×			×							
4		×				×						
5		×					×			×		
6	×		×	×				×			×	
7		×				×	×			×		
8	×	×	×	×	×	×		×	×		×	×
9	×		×									

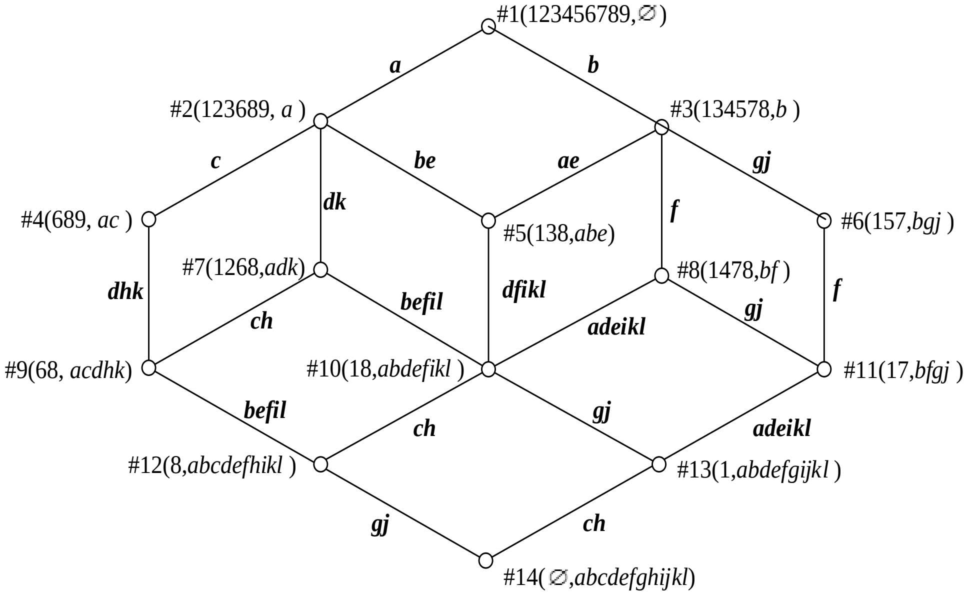

Figure 1. The Hasse Graph of Table 1

3. Theorems of Attribute Waned Value

Definition 7. Let $\mathrm{K}=(U,M,I)$ be a formal context. (X_1,Y_1), $(X_2,Y_2)\in\mathcal{B}(\mathrm{K})$, $(X_1,Y_1)\prec(X_2,Y_2)$. We call $\mathrm{Y}_1-\mathrm{Y}_2$ as an attribute waned value of $(\mathrm{X}_1,\mathrm{Y}_1)$, and denote all attribute waned values of K as $\mathcal{W}(\mathrm{K})$.

All attribute waned values in Fig. 1 are marked beside the lines.

Lemma 2. Let $\mathrm{K}=(U,M,I)$ be a formal context, each attribute waned value of $\mathcal{B}(\mathrm{K})$ is discernibility attribute set.

Proof. $\forall(X_1,Y_1)\in\mathcal{B}(\mathrm{K})$, Let A be an attribute waned value of (X_1,Y_1), (X_2,Y_2) is a direct superconcept and $A=Y_1-Y_2$. $\because$ $Y_1\supset Y_2$, $\therefore Y_1\cup Y_2=Y_1$, $Y_1\cap Y_2=Y_2$.

$\therefore$ $A=Y_1\cup Y_2-Y_1\cap Y_2=DIS((X_1,Y_1),(X_2,Y_2))$ $\therefore$ A is a discernibility attribute set.

Lemma 3. Let $\mathrm{K}=(U,M,I)$ be a formal context. Then a discernibility attribute set $DIS((A_i,B_i),(A_j,B_j))$ must be a superset of some waned value in $\mathcal{B}(\mathrm{K})$.

Proof. $\exists(X_1,Y_1),(X_2,Y_2)\in\mathcal{B}(\mathrm{K})$, the supremum of (X_1,Y_1) and (X_2,Y_2) is (X_0,Y_0). By Lemma 1, $(X_0,Y_0)=(g(Y_1\cap Y_2),Y_1\cap Y_2)$. $\because$ $Y_0=Y_1\cap Y_2\subseteq Y_1$, $\therefore$ (X_0,Y_0) is the superconcept of (X_1,Y_1), and there is a direct super concept sequence $(X_0,Y_0)\succ(A_1,B_1)\succ(A_2,B_2)\succ\cdots\succ(A_m,B_m)\succ(X_1,Y_1)$. $\therefore$ $Y_0\subset B_1\subset B_2\subset\cdots\subset B_m\subset Y_1$. $\therefore Y_1=Y_0\cup(B_1-Y_0)\cup(B_2-B_1)\cup\cdots\cup(Y_1-B_m)$. Similarly, $Y_2=Y_0\cup(B_1'-Y_0)\cup(B_2'-B_1')\ \cup\cdots\ \cup(Y_2-B_n')$.

$$\because DIS((X_1,Y_1),(X_2,Y_2))=Y_1\cup Y_2-Y_1\cap Y_2=Y_1\cup Y_2-Y_0=$$
$$\big(Y_0\cup(B_1-Y_0)\cup(B_2-B_1)\cup\cdots\cup(Y_1-B_m)\big)\bigcup$$
$$\big(Y_0\cup(B_1'-Y_0)\cup(B_2'-B_1')\cup\cdots\cup(Y_2-B_n')\big)-Y_0,$$

$\because$ Y_0 does not have the same attribute with (B_1-Y_0), (B_2-B_1), $\cdots$, (Y_1-B_m), $(B_1'-Y_0)$, $(B_2'-B_1')$, $\cdots$, (Y_2-B_n').

$$\therefore DIS((X_1,Y_1),(X_2,Y_2))=(B_1-Y_0)\cup(B_2-B_1)\cup\cdots$$

$\cup (Y_1 - B_m) \cup (B_1' - Y_0) \cup (B_2' - B_1') \cup \cdots \cup (Y_2 - B_n')$.

$\because$ $(B_1 - Y_0), (B_2 - B_1), \cdots, (Y_1 - B_m)$,

$(B_1' - Y_0), (B_2' - B_1'), \cdots, (Y_2 - B_n')$ are attribute waned values. $\therefore$ $DIS((X_1, Y_1), (X_2, Y_2))$ must be a superset of a certain waned value.

Example 2. The Hasse gragh is shown in Figure 1 , The supremum of concept #9 $(68, acdhk)$ and #11 $(17, bfgj)$ is #1$(123456789, \varnothing)$. Select any one direct superconcept sequence from #9$(68, acdhk)$ to #1$(123456789, \varnothing)$ ： #9$(68, acdhk) \prec$ #4$(689, ac) \prec$ #2$(123689, a) \prec$ #1$(123456789, \varnothing)$, and select any one direct superconcept sequence from #11 $(17, bfgj)$ to #1$(123456789, \varnothing)$: #11$(17, bfgj) \prec$ #6$(157, bgj) \prec$ #3$(134578, b) \prec$ #1$(123456789, \varnothing)$. dhk, c, a and f, gj, b are their attribute waned values respectively. $DIS(\#9, \#11) = DIS((68, acdhk), (17, bfgj)) = acdhk \cup bfgj - acdhk \cap bfgj = abcdfghjk$ is the superset of above attribute waned values.

Lemma 4[17]. Let Λ_{FC} be a discernibility attribute matrix, $H_{ij} \subseteq H_{pq}$. Then the discernibility function $F(\Lambda_{FC})$ has the same minimal disjunctive normal form as

$$F(\Lambda_{FC} - \{H_{pq}\}).$$

Lemma 5. Let $\mathrm{K} = (U, M, I)$ be a formal context. Then the discernibility function $F(\Lambda_{FC})$ has the same minimal disjunctive normal form as $F(\mathcal{W}(\mathrm{K}))$.

Proof. By Lemma 3, each H_{ij} is a superset of some attribute waned values. By Lemma 4, because $\mathcal{W}(\mathrm{K})$ is the set of all attribute waned values, the discernibility function $F(\Lambda_{FC})$ has the same minimal disjunctive normal form as $F(\mathcal{W}(\mathrm{K}))$.

4. Theorems of Infimum Irreducible Concept

Lemma 6. Let $\mathrm{K} = (U, M, I)$ be a formal context. The discernibility function $F(\mathcal{W}(\mathrm{K}))$ has the same minimal disjunctive normal form as $\mathcal{W}_0(\mathrm{K})$, where $\mathcal{W}_0(\mathrm{K})$ is an attribute waned value set of all infimum irreducible concepts.

Proof. By Lemma 4, we need prove that every attribute waned value in $\mathcal{W}(\mathrm{K}) - \mathcal{W}_0(\mathrm{K})$ is the superset of some attribute waned value in $\mathcal{W}_0(\mathrm{K})$. In other words, we need prove that every attribute waned value of the infimum non-irreducible concept is the superset of some attribute waned value of the infimum irreducible concept. $\forall (X, Y)$ is an infimum non-irreducible concept, so (X, Y) has more than one direct super concepts. Suppose (X_1, Y_1) and (X_2, Y_2) are any direct super concepts of (X, Y), and their supremum is (X_0, Y_0). We can get two direct super concept sequences: $(X_0, Y_0) \succ (A_1, B_1) \succ (A_2, B_2) \succ \cdots \succ (A_m, B_m) \succ (X_1, Y_1)$, $(X_0, Y_0) \succ (A_1', B_1') \succ (A_2', B_2') \succ \cdots \succ (A_n', B_n') \succ (X_2, Y_2)$. $\because$ (X_1, Y_1) and (X_2, Y_2) are the direct super concepts of (X, Y) . $\therefore$ $Y_0 \subset B_1 \subset B_2 \subset \cdots \subset B_m \subset Y_1 \subset Y$, $Y_0 \subset B_1' \subset B_2' \subset \cdots \subset B_n' \subset Y_2 \subset Y$.

$\therefore$ $Y_0 \cup (B_1 - Y_0) \cup (B_2 - B_1) \cup \cdots \cup (B_m - B_{m-1}) \cup (Y_1 - B_m) \cup (Y - Y_1) = Y$ （1）

$Y_0 \cup (B_1' - Y_0) \cup (B_2' - B_1) \cup \cdots \cup (B_n' - B_{n-1}') \cup (Y_2 - B_n') \cup (Y - Y_2) = Y$ （2）

$\because$ Y_0 has no common element with both

$(B_1 - Y_0) \cup (B_2 - B_1) \cup \cdots \cup (B_m - B_{m-1}) \cup (Y_1 - B_m) \cup (Y - Y_1)$

And

$$(B_1' - Y_0) \cup (B_2' - B_1) \cup \cdots \cup (B_n' - B_{n-1}') \cup (Y_2 - B_n') \cup (Y - Y_2).$$

4 Bψ (1) (2),

$$(B_1-Y_0)\cup(B_2-B_1)\cup\cdots\cup(B_m-B_{m-1})\cup(Y_1-B_m)\cup(Y-Y_1)=$$
$$(B_1'-Y_0)\cup(B_2'-B_1)\cup\cdots\cup(B_n'-B_{n-1}')\cup(Y_2-B_n')\cup(Y-Y_2) \qquad (3)$$

And ∵ $Y_0=Y_1\cap Y_2$.

∴ $(B_1-Y_0)\cup(B_2-B_1)\cup\cdots\cup(B_m-B_{m-1})\cup(Y_1-B_m)$ has no common element with $(B_1'-Y_0)\cup(B_2'-B_1)\cup\cdots\cup(B_n'-B_{n-1}')\cup(Y_2-B_n')$.

∴ $(B_1-Y_0)\cup(B_2-B_1)\cup\cdots\cup(B_m-B_{m-1})\cup(Y_1-B_m)\subseteq(Y-Y_2)$ ∴ $(Y_1-B_m)\subseteq(Y-Y_2)$.

Similarly, $(B_1'-Y_0)\cup(B_2'-B_1)\cup\cdots\cup(B_n'-B_{n-1}')\cup(Y_2-B_n')\subseteq Y-Y_1$. ∴ $(Y_2-B_n')\subseteq(Y-Y_1)$. By the above deduction, attribute waned values of (X,Y) must be the superset of some waned value which belongs to a direct super concept. If a direct super concept is an infimum irreducible concept, then the attribute waned value set of (X,Y) must be the superset of its attribute waned value set. In other words, an infimum non-irreducible concept waned value set is the superset of an infimum irreducible concept waned value set.

Example 3. A given concept lattice is shown in Figure 1, the concept #13 is an infimum non-irreducible concept, and has two direct super concepts: the concept #10 and the concept #11. By Lemma 6, its attribute waned values must be the superset of one direct superconcept . #13 has two attribute waned values:*gj* and *adeikl*. Among them, *gj* is the superset of the #11 attribute waned value:*gj*, and *adeikl* is the superset of the #10 atrribute waned value: *adeikl*. Both #10 and #11 are not infimum irreducible concepts, so their attribute waned values must be the superset of some superconcept waned value respectively. Amongs them, *gj* is the superset of the #6 attribute waned value: *gj*, and *adeike* is the superset of the #5 attribute waned value: *ae*. The concept #6 is an infimum irreducible concept, so the waned value *gj* of #13 is the superset of the attribute waned value *gj* which belongs to #6.

By Lemma 6, we just need get the minimal disjunctive normal form of $F(\mathcal{W}_0(\mathbb{K}))$, which belongs to the set of infimum irreducible concepts $\mathcal{W}_0(\mathbb{K})$

Lemma 7. If $(X,Y)\in\mathfrak{B}(\mathbb{K})$ be an infimum irreducible concept, then (X, Y) is an attribute concept, and its attribute waned value has no common element with the other attribute concepts.

Proof. Let (X,Y) be an infimum irreducible concept, and (X_0,Y_0) the only direct super concept of (X,Y), $Y\text{-}Y_0=\{m_1, m_2, \ldots, m_k\}$. ∵$\{m_i\}\subseteq Y$ ∴ By the property 1(6), $g(m_i)\supseteq g(Y)\quad(i=1,2,\cdots,k)$. If $g(m_i)\supset g(Y)$, then $(g(m_i), f(g(m_i)))$ is the super concept of (X, Y). ∵ (X_0, Y_0) is the direct super concept, ∴ $g(m_i)\supseteq g(Y_0)$,∴ $f(g(m_i))\subseteq f(g(Y_0))$,∵ $m_i\in f(g(m_i))\quad(i=1,2,\cdots,k)$ and $f(g(Y_0))=Y_0$, ∴ $m_i\in Y_0$. The conclusion contradicts with $Y\text{-}Y_0=\{m_1, m_2, \ldots, m_k\}$. ∴ $g(m_i)=g(Y)\quad(i=1,2,\cdots,k)$.∵ $g(m_i)=X$ and $f(g(m_i))=f(g(Y))=Y$. ∴ $(X,Y)=(g(m_i), f(g(m_i)))$. ∴ (X,Y) is an attribute concept.

Let $(X1,Y1)$ and $(X2,Y2)$ be two different infimum irreducible concepts, and attribute waned values are $\{m_1,m_2,\cdots,m_k\}$ and $\{m_1',m_2',\cdots,m_t'\}$ respectively. We can get the equations $g(m_1)=g(m_2)=\cdots=g(m_k)=X_1$ and $g(m_1')=g(m_2')=\cdots=g(m_t')=X_2$. ∵ $X_1\neq X_2$, ∴ $\{m_1,m_2,\cdots,m_k\}$ has no common element with $\{m_1',m_2',\cdots,m_t'\}$.

Lemma 8. If $g(m)\neq g(\{m'\in M\mid g(m')\supset g(m)\})\ \ m\in M$, then the attribute concept $(g(m), f(g(m)))$ is an infimum irreducible concept.

Proof. $Y=\{m'\in M \mid g(m')\supset g(m)\}$, by the property 1（4）, $g(Y)=g(\{m' \mid g(m')\supset g(m)\})= \cap\{g(m') \mid g(m')\supset g(m)\}$ $\because$ $g(m')\supset g(m)$, $\therefore$ $\left(g(m')\bigcap g(m)\right)\supset g(m)$ $\therefore$ $g(Y)\supset g(m)$. Suppose $\exists Y'$, $g(Y)\supset g(Y')\supset g(m)$. $\because$ $\forall m'\in Y'$, $g(m')\supseteq g(Y')$. $\therefore$ $g(m')\supset g(m)$ $\because$ $Y=\{m'\in M \mid g(m')\supset g(m)\}$, $\therefore$ $Y'\subseteq Y$. $\therefore$ $g(Y')\supseteq g(Y)$. That contradicts with $g(Y)\supset g(Y')$, thus there is no Y' to make $g(Y)\supset g(Y')\supset g(m)$ true. So $(g(Y), f(g(Y))$ is the direct super concept of $(g(m), f(g(m)))$. Suppose $\exists(X'',Y'')$, (X'',Y'') is the direct super concept of $(g(m), f(g(m)))$, and $X''\neq g(Y)$. Then $g(Y'')\supset g(m)$ $\therefore$ $g(m'')\supseteq g(Y'')\supset g(m)$, $\forall m''\in$ Y''. Similarly, $\because$ $Y=\{m'\in M \mid g(m')\supset g(m)\}$. $\therefore$ $m''\in Y$ $\therefore$ $Y''\subseteq Y$ $\therefore$ $g(Y'')\supseteq g(Y)$ $\because$ Both (X'',Y'') and $(g(Y), f(g(Y))$ are the super concepts of $(g(m), f(g(m)))$, $\therefore$ $g(Y'')\supset g(Y)$ is incorrect, and $g(Y'')=g(Y)$ is true. The conclusion is contradicts with $X''\neq g(Y)$. Therefore, $(g(m), f(g(m)))$ is an infimum irreducible concept.

We can get following conclusions by the lemmas.

Theorem 1. Let $\mathrm{K}=(U,M,I)$ be a formal context. $\mathcal{B}_0(\mathrm{K})$ is the set of all infimum irreducible concepts. That select any one element from each attribute waned value of the element of $\mathcal{B}_0(\mathrm{K})$ can form an attribute reduction set.

Proof. By [4], the set X is an attribute reduction of K, if and only if X contains all attributes of any one conjunctive term of $F(\Lambda_{FC})$, where discernibility function $F(\Lambda_{FC})$ is a discernibility function which is converted to minimal disjunctive normal form. By Lemma 5, $F(\Lambda_{FC})$ and $F(\mathcal{W}(\mathrm{K}))$ have the same minimal disjunctive normal form. By Lemma 6, $F(\mathcal{W}(\mathrm{K}))$ and $F(\mathcal{W}_0(\mathrm{K}))$ have the same minimal disjunctive normal form. By Lemma 7 and Lemma 8, the attribute concept $(g(m), f(g(m)))$ is an infimum irreducible concept, if and only if $g(m)\neq g(\{m'\in M \mid g(m')\supset g(m)\})$, $m\in M$. For the set of the attribute waned value of all infimum irreducible concepts $\mathcal{W}_0(\mathrm{K})$, each attribute waned value has no common element with the other attribute waned values of attribute concepts. Therefore, that select any one element from each attribute waned value of the element of $\mathcal{B}_0(\mathrm{K})$ can form an attribute reduction.

Definition 8[4]. Let all reductions of the formal context $\mathrm{K}=(U,M,I)$ be $\{D_i \mid D_i\ \textit{is a reduction},\ i\in\tau\}$ (τ is an index set). M is made up of three parts:(1) core attributes b: $b\in\bigcap_{i\in\tau}D_i$. (2) relative necessary attributes c: $c\in\bigcup_{i\in\tau}D_i-\bigcap_{i\in\tau}D_i$.(3) unnecessary attributes d: $d\in M-\bigcap_{i\in\tau}D_i$.

If the attribute waned value of an infimum irreducible concept only has one attribute, the attribute is called as “core attribute”. If the attribute waned value of an infimum irreducible concept has more than one attributes $m1$,$m2$,…,mk, and $g(m1)$=$g(m2)$=…=$g(m$k), the attributes are called as “relative necessary attribute”. If the attribute concept is not an infimum irreducible concept, the attribute is called as “unnecessary attributes”.

Example 4. In Figure 1, concepts #2, #3, #4, #6, #7, #8 are infimum irreducible concepts. select any one element from each attribute waned value of these concepts, and then form an attribute reduction set. All attribute reduction set of this examle are {*a*, *b*, *c*, *d*, *f*, *g*}, {*a*, *b*, *c*, *k*, *f*, *g*},{*a*, *b*, *c*, *d*, *f*, *j*}. In concepts #2, #3, #4, #6, #7, #8, some attribute waned values have one attribute, then those attributes are core attributes, such as *a*, *b*, *c*, *f*, some attribute waned values have more than one attributes, then those attributes are relative necessary attributes, such as *d*, *k*, *g*, *j*, and the attributes that do not appear in infimum irreducible concepts are unnecessary attributes, such as *e*, *i*, *l*.

5. Attribute Reduction Algorithm

By Theorem 1, we present a new attribute reduction algorithm.

Input: Formal context $\mathbb{K}=(U,M,I)$,where $M=\{m_1,\cdots,m_q\}$.

Output: Attribute reductions of $\mathbb{K}$.

Step: Declare arrays: $A[1:q]$, $B[1:q]$, $C[1:q]$, where elements of $A[i]$ are $\{m_j \in M \mid g(m_j) \supset g(m_i)\}$, elements of $B[i]$ are $\{m_j \in M \mid g(m_j) = g(m_i)\}$, and $C[i]=yes \mid no$ (yes means that m_i is deleted, no means that m_i is un-deleted);

$For\ i=1\ to\ q \quad A[i]:=\varnothing \quad B[i]:=\varnothing \quad C[i]:=no \quad End\ For$ /*initialization

$For\ i=1\ to\ q$

$For\ \ j=1\ to\ q$

$If\ \ g(m_j)\supset g(m_i)\ Then\ A[i]:=A[i]\cup\{m_j\}$

$If\ \ g(m_j)=g(m_i)\ Then\ B[i]:=B\{i]\cup\{m_j\}$

$End\ For$

$End\ For$

$For\ i=1\ to\ q$

$If\ \ g(m_i)=g(A[i])\ \ Then\ \ C[i]:=yes$

By Lemma 8, the attribute concept of m_i is not an infimum irreducible concept.*/

$End\ For$

$For\ i=1\ to\ q-1$

$For\ \ j=i+1\ to\ q$

$If\ \ B[i]=B[j] \quad Then\ \ C[i]:=yes$

$End\ For$

$End\ For$

$S:=\varnothing$

$For\ i=1\ to\ q$

$If\ \ C[i]=no \quad Then\ \ S:=S\cup\{B[i]\}$

$End\ For$

$Return \quad \times S$ /*Compute the Cartesian product of S to get all attribute reductions

Example 5. The formal context is shown in Table 2 (it is the same as the table of [4]), where $m_1=a$, $m_2=b$, $m_3=c$, $m_4=d$, $m_5=e$, is reduced as follows.

Table 2. A Given Formal Context

	a	b	c	d	e
1	×	×		×	×
2	×	×	×		
3				×	
4	×	×	×		

Execute steps（2）-（7） of the algorithm：

$A[1]=\varnothing$, $A[2]=\varnothing$, $A[3]=\{a,b\}$, $A[4]=\varnothing$, $A[5]=\{a,b,d\}$

$B[1]=\{a,b\}$, $B[2]=\{a,b\}$, $B[3]=\{c\}$, $B[4]=\{d\}$, $B[5]=\{e\}$

Execute steps（8）-（10）:
$C[1]=no$, $C[2]=no$, $C[3]=no$, $C[4]=no$, $C[5]=yes$,
Execute steps（11）-（15）:
$C[1]=yes$, $C[2]=no$, $C[3]=no$, $C[4]=no$, $C[5]=yes$,
Execute steps（16）-（19）:
$S=\{\{a,b\},\{c\},\{d\}\}$
Execute step（20）:
$\{a,c,d\},\{b,c,d\}$
All attribute reductions are got.

Theorem 2. The time complexity of algorithm is $O(q^2k)$, where k is the number of objects, and q is the number of attributes.

Proof. The time complexity of the algorithm step (1) is $O(qk)$, the time complexity of the algorithm steps（2）-（7） is $O(q^2k)$, the time complexity of the algorithm steps（8）-（10）is $O(qk)$, the time complexity of the algorithm steps（11）-（15）is $O(q^2k)$, the time complexity of the algorithm steps（16）-（19）is $O(qk)$, and the time complexity of step（20）is $O(q^2k)$. So the time complexity of the whole algorithm is $O(q^2k)$.

Theorem 3. The space complexity of algorithm is $O(q^3+3q^2+(k+2)q)$, where k is the number of objects, and q is the number of attributes.

Proof. Let $\mathbb{K}=(U,M,I)$ be a formal context, $U=\{u_1,u_2,\cdots,u_k\}$, $M=\{m_1,m_2,\cdots,m_q\}$, $T[1:q]$ is an array of k bits binary number. If j-th object u_j has i-th attribute m_i, the j-th bit of $T[i]$ is "1", and j-th object u_j does not have i-th attribute m_i, then the j-th bit of $T[i]$ is "0". Meanwhile, the i-th binary number $T[i]$ denotes $g(m_i)$. The formal context array $T[1:q]$ needs qk *bits* space. Similarly, $A[1:q]$ has q q-bit binary numbers, and it needs q^2 *bits* space. $B[1:q]$ has q q-bit binary numbers, and it needs q^2 bits space. $C[1:q]$ has q 1-bit binary numbers, and it needs q-*bits* space. $S[1:q]$ has q q-bit binary numbers, and it needs q^2 *bits* space. Cartesian product of S has q^2 q-bit binary numbers, and it needs q^3 *bits* spaces. So, the step (1) needs qk bits, the steps（2）-（7） need $2q^2$ bits; the steps（8）-（10） need $2q$ bits, the steps（11）-（15）do not need space, the steps（16）-（19）need q^2 bits at most, the step（20）needs q^3 bits at most, and the whole space are $q^3+3q^2+(k+2)q$ bits.

Example 6. For the formal context of Example 1, Suppose $m_1=a,m_2=b,m_3=c,m_4=d$, $m_5=e,m_6=f,m_7=g,m_8=h$, $m_9=i,m_{10}=j,m_{11}=k,m_{12}=l$, $u_1=1,u_2=2,u_3=3,u_4=4$, $u_5=5,u_6=6,u_7=7,u_8=8,u_9=9$

After Executing step(20), the ouputs are 111101100000, 111001100010, 111101000100, 111001000110, and they conform to the rule that use at most 12^2 12-bit binary numbers to get result.

6. Conclusion

This paper presents a new concept reduction method, which selects elements from each infimum irreducible concept. Then we prove that infimum irreducible concepts must be attribute concepts.

Dually, all conclusions in this paper can convert to the object reduction method and further get the method that can reduce both objects and attributes.

The presented algorithms maybe have some methods to simplify the steps. So there are requirements for the further research.

Acknowledgements

This research has been supported by the National Natural Science Foundation of China (NSFC Grant No. 61273019) and the Youth Fund of University of Science and Technology Liaoning (Grant No. 2014QN21).

References

[1] R. Wille, "Restructuring lattice theory: an approach based on hierarchies of concepts", International Conference on Formal Concept Analysi, **(1982)**.
[2] D. Li, C. Tian, B. Li, G. Sun and J. Kang, "Network Traffic Classification Based on Subspace Clustering-based Method", Journal of Harbin University of Science and Technology, vol. 20, no. 2, **(2015)**, pp. 63-68.
[3] A. Gély, Y. Renaud, R. Medina and L. Nourine, "Uncovering and reducing hidden combinatorics in Guigues-Duquenne bases", Lecture Notes in Computer Science, **(2005)**, pp. 235-248.
[4] W. Zhang, L. Wei and J. Qi, "Attribute reduction theory and approach to concept lattice", Science in China, vol. 48, no. 6, **(2005)**, pp. 713-726.
[5] X. Wang and J. Ma, "A novel approach to attribute reduction in concept lattices", Lecture Notes in Computer Science, **(2006)**, pp.522-529.
[6] K. S. K. Cheung and D. Vogel, "Complexity reduction in lattice-based information retrieval", Information Retrieval, vol. 8, no. 2, **(2005)**, pp. 285-299.
[7] C. Carpineto and G. Romano, "Using concept lattices for text retrieval and mining", Formal Concept Analysis State of the Art Proc of the First International Conference on Formal Concept Analysis, **(2005)**, pp. 161-179.
[8] M. Shao, L. Guo and L. Li, "in A novel attribute reduction approach based on the object oriented concept lattice", Springer Berlin Heidelberg, **(2011)**, pp. 71-80.
[9] L. Wei and A. Pan, "Attribute characteristics of object oriented concept lattices based on the join-irreducible elements", 2012 IEEE International Conference on Granular Com, **(2012)**.
[10] C. Aswanikumar and S. Srinivas, "Concept lattice reduction using fuzzy K-Means clustering", Expert Systems With Applications, vol. 37, no. 3, **(2010)**, pp.2696–2704.
[11] J.S. Mi, Y. Leung and W.Z. Wu, "Approaches to attribute reduction in concept lattices induced by axialities", Knowledge-Based Systems, vol. 23, no. 6, **(2010)**, pp.504-511.
[12] Q. Wan and L. Wei, "Approximate concepts acquisition based on formal contexts", Knowledge-Based Systems, **(2014)**.
[13] D. Pei and J.S. Mi, "Attribute reduction in decision formal context based on homomorphism", International Journal of Machine Learning & Cybern, vol. 2, no. 4, **(2011)**, pp. 289-293.
[14] J.-M. Ma, Y. Leung and W.-X. Zhang, "Attribute reductions in object-oriented concept lattices", International Journal of Machine Learning and Cybernetics, **(2014)**.
[15] L. Wei and Q. Li, "Covering-based reduction of object-oriented concept lattices", Lecture Notes in Computer Science, 6954 LNAI, **(2011)**, pp. 728 – 733.
[16] B. Ganter, R. Wille and C. Franzke, "Formal concept analysis: mathematical foundations", 2009 3rd IEEE International Conference on Digital, **(1998)**.
[17] R. Zhao, H. Zhang, L.I. Cuiling, L.U. Jianfeng and J. Wang, "Disjunctive normal form Generation algorithm for discernibility function in rough set theory", Computer Engineering, vol. 32, no. 2, **(2006)**, pp.183-185.

