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Model of Torsional Waves Based on Transverse and Layered One-Dimensional Component of Infinite Length

Wu Di[1,2] , Kang Wei-Xin[1],Wang Hong-Ru[1] and Liu Yu-Mei[1]

[1]*College of Information and Communication Engineering, Harbin Engineering University, Harbin, 150001, China*
[2]*College of Computer and Information Engineering, Heilongjiang University of Science and Technology, 150024, China*
wudi55dd@sina.com；Kangwx@126.com

Abstract

The dispersion of torsional wave in transverse and layered one-dimensional component of infinite length is investigated. The study is carried out by treating the one-dimensional component as a cylinder of infinite length firstly, Then put forward a way that the component is divided into a multilayer to analyze torsional wave propagation, the model is constructed in cylindrical coordinates for homogeneous and layered component, the governing equation of torsional wave was derived based on one-dimensional elastic theory and Whittaker's function, obtain harmonic wave solution of the governing equation by making use of the boundary and continuity conditions of adjacent layers. Finally the result of numerical simulation illustrated the influence of component's density and shear module on torsional angels and phase velocity. It has been observed the velocity is changing as the change of parameters of shear module and density according to the curve. The conclusion made shown the consistency with the classical theory, and the effective on model such as bearings, engineering mechanics and displacement of particles.

Keywords*: the torsional wave, governing equation, one-dimensional component, numerical simulation, Whittaker's function*

1. Introduction

More attention are paid to the dispersion of torsional wave in half space ,and less attention is paid on the one-dimensional component based on the torsional wave how to model and get the solutions to the govening equtions. S.Xie and K.Liu [1] studied the transversely isotropic tube propagation of transient torsional wave,they adopted the means that the top of the pile under a torque is applied to the surface of a sudden ,putting into use variable separation method and Laplace transform,the conclusion that discontinuous distribution of shear stress and shear stress anisotropy with the tube material distribution were drawn. J.Kudlicka [2-3]further studied the dispersion of torsional wave in transversely isotropic thick-walled cylinder of infinite length ,obtained transmission amplitude and phase velocity curves to provide a reference for the study of the anisotropy of the shell structure on torsional wave .T.Yokoyama and K.Liu [4] also studied the distribution of shear stress of torsional wave for strengthening rod from transient torsional wave analyzed and the experimental data analyzed. Q.Wang[5] presents the wave propagation in a cylinder coated with a thin piezoelectric layer. The decoupled torsional wave velocity and the dispersion curves for the two- mode shell model are obtained. Z.C.XI et al. [6] proposed a layered element method for analyzing frequency and group velocity dispersive behaviours of waves in a laminated composite cylinder surrounded by a fluid. The method applied finite elements to model the radial displacement of the cylinder and the radial pressure of the fluid, and

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complex exponentials to express the axial and circumferential displacements of the cylinder as well as the axial and tangential pressures of the fluid. Tamer Kepceler[7] researched the dispersion of the torsional wave propagation in the finitely prestrained bi-material compounded cylinder,and prove the mechanical relations of the materials of the components of the cylinder are described by the harmonic potential.

S.D. Akbarov et al. [8] studied torsional wave dispersion in a three-layered (sandwich) hollow cylinder with finite initial strains. The investigations were carried out within the scope of the piecewise homogeneous body model with the use of the three-dimensional linearized theory of elastic waves in initially stressed bodies. X. HAN and D. XU.[9] proposed a method for analyzing transient waves in cylindrical shells of a functionally graded material (FGM) excited by impact point loads. In the present method, the FGM shell is divided into layer elements with three nodal lines along the wall thickness. The material property within each element is assumed to vary linearly in the thickness direction, which represented the spatial variation of material property of FGM. Michael El-Raheb [10] studied the effect on transient waves of circumferential and radial inhomogeneity were studied in a plane-strain hollow cylinder. A periodic circumferential inhomogeneity modulating a constant value was analyzed adopting the Galerkin method where trial functions were chosen as the axisymmetric and asymmetric modes of the homogeneous cylinder.

Kuihua Wang and Zhiqing Zhang [11] investigated the torsional vibration of an end bearing pile embedded in a homogeneous poroelastic medium and subjected to a time-harmonic torsional loading. By using the separation of variables technique, the torsional response of the soil layer was solved first, then based on perfect contact between the pile and soil, the dynamic response of the pile is obtained in a closed form. Yuanqiang Cai et al. [12] were Based on the Muki’s model, studied the response of a cylindrical elastic pile embedded in a homogeneous poroelastic medium and subjected to torsional loading. Guocai Wang et al.[13-14] researched the dynamic response of single piles vertically embedded in the saturated half space and subjected to harmonic torsional loadings From the axisymmetrical point of view, by means of integral transform and Muki’s methods for the first time. A. Chattopadhyay. et al.[15] in his paper presented propagation of torsional waves in an inhomogeneous layer over an inhomogeneous half-space . S. Gupta[16] in his studies,presented the dispersion equation which determines the velocity of torsional surface waves in a homogeneous layer of finite thickness over an initially stressed heterogeneous half-space.

In all,many researchers focused on the piles of embedded in satured soil and torsional surface wave in half space, others studied the hollow cylinder. Attentions on one-dimensional component of thin or infinite length were less. The aim of the present paper is to investigate axisymmetric torsional wave propagation in transverse and layered one-dimensional component of infinite length ,construct the model and dispersion equation, obtain the solution of equation and so on, to find out the regulations between the density, module of the material of component and dispersion curve .

2. Construction of Governing Equations

The problem studied in the paper is the dispersion of torsional waves of one-dimensional component of infinite length. It is considered to use a solid cylinder of infinite length to simulate the one-dimensional component. The model is shown in Figure 1. The main assumptions adopted in the paper are :(1)one-dimensional component’s material is isotropic, homogeneous and linearly elastic;(2)The deformations and rotations are small and omitted.(3) one-dimensional component is subjected to an impulsively applied torque at the surface $z=-H$ (Figure 1). Due to the symmetry of axis, so it is considered wave propagation in the plane $\theta=0$.

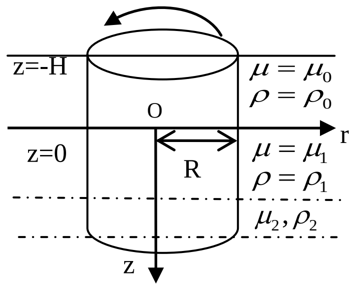

Figure 1. Model of One-Dimensional Component

Constitutive equation and strain displacement relations using a polar cylindrical coordinate system (r,θ,z) ,where the origin O is located at the cylinder center of the interface of two layer is set $z=0$,and the cylinder axis is taken to be the z-axis, and denoting the corresponding displacement components by (u_r,u_θ,u_z) ,for torsional wave ,the displacement field is valid, u_r and u_z are zero.($u_r=0, u_z=0$), strain-displacement relations are shown in the paper[3] ,the equation of torsional motion may be written as:

$$\frac{\partial \tau_{r\theta}}{\partial r}+\frac{\partial \tau_{z\theta}}{\partial z}+\frac{2\tau_{r\theta}}{r}=\rho\frac{\partial^2 u_\theta}{\partial t^2} \tag{1}$$

Where $\frac{\partial u_\theta}{\partial t}=v_\theta$ is the circumferential particle velocity, $\tau_{r\theta}$ and $\tau_{z\theta}$ is the shear stress component, ρ is the mass density and t is the time.

The strain rate-particle velocity relations and stress-strain relations， respectively, in terms of cylindrical coordinate system are as followed[3]:

$$\tau_{r\theta}=\mu(z)\gamma_{r\theta},\ \tau_{z\theta}=\mu(z)\gamma_{z\theta}$$
$$\frac{\partial \gamma_{r\theta}}{\partial t}=\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r},\frac{\partial \gamma_{z\theta}}{\partial t}=\frac{\partial v_\theta}{\partial z} \tag{2}$$

Where μ is the shear modulus, r is a representative length (equal to the radius of bar) .Eliminating the shear strains ($\gamma_{r\theta},\gamma_{z\theta}$), leads to a system of the governing equations in dimensionless form.

Now, Using (1) in (2), the only non-vanishing equation of motion [16] is given by

$$\frac{\partial^2 u_\theta}{\partial r^2}+\frac{\partial^2 u_\theta}{\partial z^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2}+\frac{\mu'}{\mu}\frac{\partial u_\theta}{\partial z}=\frac{\rho}{\mu}\frac{\partial^2 u_\theta}{\partial t^2} \tag{3}$$

Where $\mu'=\frac{d\mu}{dz}$.We considerate the harmonic wave solution of Eq.(3) is the form

$$u_\theta=V(z)J_1(Kr)e^{iwt} \tag{4}$$

Using Eq. (4) in Eq. (3), then Eq.(3) reduces to the form

$$\frac{d^2V(z)}{dz^2}+\frac{\mu'}{\mu}\frac{dV(z)}{dz}-K^2(1-\frac{c^2}{c_s^2})V(z)=0 \tag{5}$$

Where $c=\frac{\omega}{K}$, is the torsional wave velocity , $c_s=\sqrt{\frac{\mu}{\rho}}$, ω is the angular frequency , K is the angular wave number, and $J_1(Kr)$ is the first order Bessel function of first kind.

3. Boundary Conditions and Solution of Layer

According to boundarty and continous conditions of Figure 1, we get the forms as follows:

(1) Initial conditions $t = 0$ and at the top $z = \text{-H}$.

$$u_{\theta 0|t=0} = 0, \frac{\partial u_{\theta 0}}{\partial t}|_{t=0} = 0 \text{ , } \frac{\partial u_{\theta 0}}{\partial z}|_{z=-H} = 0 \text{ .} \tag{6a}$$

(2) On the junction $z = 0$, the continuity of the stress and displacement requires that

$$\mu_0 \frac{\partial u_{\theta 0}}{\partial z}|_{z=0} = \mu_1 \frac{\partial u_{\theta 1}}{\partial z}|_{z=0} \text{ , } u_{\theta 0} = u_{\theta 1} \tag{6b}$$

(3) Displacement is limited when $z \to \infty$ as

$$\lim_{z \to \infty} u_{\theta 1}(z) = 0 \tag{6c}$$

Where $u_{\theta 0}$ and $u_{\theta 1}$ are the displacement in the line of $z = 0$.

In the section, according to the characters of material of one-dimensional component, two cases were considered to resolve each solution, homogeneous and layered component, respectively.

Case I: Assuming the upper layers of the one-dimensional component satisfy the conditions,

$$\mu_0 = G, \rho_0 = \rho \tag{7}$$

So using Eq. (7) in Eq.(5),we get

$$\frac{d^2 V(z)}{dz^2} - K^2 (1 - \frac{c^2}{c_s^2}) V(z) = 0 \tag{8}$$

Where $c_s = \sqrt{\frac{G}{\rho}}$,So the solution of Eq.(9) is the form as follow:

$$V(z) = A_1 e^{(1-\frac{c^2}{c_s^2})^{\frac{1}{2}} kz} + A_2 e^{-(1-\frac{c^2}{c_s^2})^{\frac{1}{2}} kz} \tag{9}$$

Where A_1 and A_2 are arbitrary constants and hence the displacement in the model is

$$u_{\theta 0} = (A_1 e^{(1-\frac{c^2}{c_s^2})^{\frac{1}{2}} kz} + A_2 e^{-(1-\frac{c^2}{c_s^2})^{\frac{1}{2}} kz}) J_1(kr) e^{iwt} \tag{10}$$

Case II: At $z = 0$, we assume the formulation is met:

$$\mu = \mu_1 (1 + \alpha z), \rho = \rho_1 (1 + \beta z) \tag{11}$$

Using of the eq. (3) and (4) of stress-strain relations, using Eq. (11) in Eq. (1) and eq. (5) for the bottom of $z = 0$, we obtain the form:

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \frac{\alpha}{1 + \alpha z}\frac{\partial u_\theta}{\partial z} = \frac{\rho_1}{\mu_1}\frac{(1 + \beta z)\partial^2 u_\theta}{(1 + \alpha z)\partial t^2} \tag{12}$$

$$\frac{d^2 V_1(z)}{dz^2} + \frac{\mu_1 \alpha}{\mu_1 (1 + \alpha z)}\frac{dV_1(z)}{dz} - K^2 (1 - \frac{c^2 (1 + \beta z)}{c_1^2 (1 + \alpha z)}) V_1(z) = 0 \tag{13}$$

Where $c = \frac{\omega}{K}$ the phase velocity of torsional wave is, $c_1 = \sqrt{\mu_1 / \rho_1}$ is the shear wave velocity.

We apply the Mathematics Methods, substituting $V_1(z)=\frac{\phi(z)}{(1+\alpha z)^{1/2}}$ into eq. (15), and eliminate the term $\frac{dV_1}{dz}$, we get

$$\frac{d^2\phi(z)}{dz^2}+\left(\frac{\alpha^2}{4(1+\alpha z)}-K^2\left(1-\frac{c^2(1+\beta z)}{c_1^2(1+\alpha z)}\right)\right)\phi(z)=0 \tag{14}$$

It is mathematically convenient at this stage to introduce the dimensionless quantities

$$\eta_1=\sqrt{1-\frac{c^2}{c_1^2}\frac{\beta}{\alpha}}\ ,\ \eta_2=\frac{2\eta_1 K(1+\alpha z)}{\alpha}$$

in eq.(14),the equation reduces to

$$\frac{d^2\phi}{d\eta_2{}^2}+\left(\frac{1}{4\eta_2^2}-\frac{1}{4}+\frac{s}{2\eta_2}\right)\psi(\eta_2)=0 \tag{15}$$

Where $s=\frac{\omega^2(\alpha-\beta)}{c_1\eta_1\alpha^2 K}$,Eq.(15) is the well known Whittaker's equation.

Equation (9) has boundary, so the solution of Whittaker's equation (14) is given by

$$\phi(\eta_2)=D_1W_{\frac{s}{2},0}(\eta_2)+D_2W_{\frac{-s}{2},0}(-\eta_2) \tag{16}$$

Where D_1 and D_2 are the arbitrary constants .The solution eq. (16) satisfying the condition that $\lim_{z\to\infty}V_1(z)\to 0$ *i.e.* $\lim_{\eta_2\to\infty}\phi(\eta_2)\to 0$,so may be taken as $\phi(\eta_2)=D_1W_{\frac{s}{2},0}(\eta_2)$.

Hence, $V_1(z)=D_1W_{\frac{s}{2},0}\left(\frac{2\eta_1 K(1+\alpha z)}{\alpha}\right)$, the displacement for the torsional wave in the layer is

$$u_{\theta 1}=\frac{D_1W_{\frac{s}{2},0}\left(\frac{2\eta_1 K(1+\alpha z)}{\alpha}\right)}{(1+\alpha z)^{\frac{1}{2}}}J_1(Kr)\mathrm{e}^{iwt} \tag{17}$$

Expanding the Whittaker function up to linear term eq. (17) takes the form (18).

$$u_{\theta 1}=\frac{D_1J_1(Kr)\mathrm{e}^{iwt}}{(1+\alpha z)^{\frac{1}{2}}}e^{\frac{-\eta_1 K(1+\alpha z)}{\alpha}}\left[\frac{2\eta_1 K(1+\alpha z)}{\alpha}\right]^{1/2}\left[1+\frac{\eta_1 K(1+\alpha z)}{\alpha}-s\frac{\eta_1 K(1+\alpha z)}{\alpha}\right] \tag{18}$$

According to the boundary conditions (6a),(6b),and (6c),we get (19)-(21).

$$A_1\exp\left(-\sqrt{1-\frac{c^2}{c_s^2}}KH\right)-A_2\exp\left(\sqrt{1-\frac{c^2}{c_s^2}}KH\right)=0 \tag{19}$$

$$A_1-A_2-K_1D_1=0 \tag{20}$$

$$A_1+A_2-K_2D_1=0 \tag{21}$$

Where

$$K_1=\eta_1 K\frac{\mu_1}{\mu_0}\sqrt{\frac{2\eta_1 K}{\alpha}}\left(\frac{s\eta_1 K-\eta_1 K-\alpha s}{\alpha}\right)e^{\frac{-\eta_1 K}{\alpha}}\ K_2=\sqrt{\frac{2\eta_1 K}{\alpha}}\left(1+\frac{\eta_1 K-s\eta_1 K}{\alpha}\right)e^{\frac{-\eta_1 K}{\alpha}}.$$

Eliminating A_1, A_2 and D_1 from eq.(19)-(21),we obtain

$$\begin{vmatrix} \exp(-\sqrt{1-\frac{c^2}{c_s^2}}KH) & -\exp(\sqrt{1-\frac{c^2}{c_s^2}}KH & 0 \\ 1 & -1 & -K_1 \\ 1 & -1 & -K_2 \end{vmatrix} = 0$$

Using K_1, K_2 in above determinant, we obtain

$$\eta_1 K \frac{\mu_1}{\mu_0}\sqrt{\frac{2\eta_1 K}{\alpha}}(\frac{s\eta_1 K-\eta_1 K-\alpha s}{\alpha})\exp(-\sqrt{1-\frac{c^2}{c_s^2}}KH)+\sqrt{\frac{2\eta_1 K}{\alpha}}(1+\frac{\eta_1 K-s\eta_1 K}{\alpha})\exp(-\sqrt{1-\frac{c^2}{c_s^2}}KH)$$
$$+\eta_1 K \frac{\mu_1}{\mu_0}\sqrt{\frac{2\eta_1 K}{\alpha}}(\frac{s\eta_1 K-\eta_1 K-\alpha s}{\alpha})\exp(\sqrt{1-\frac{c^2}{c_s^2}}KH)-\sqrt{\frac{2\eta_1 K}{\alpha}}(1+\frac{\eta_1 K-s\eta_1 K}{\alpha})\exp(\sqrt{1-\frac{c^2}{c_s^2}}KH)=0 \quad (22)$$

Particular Case: No matter case I or case II, when $\alpha \to 0, \beta \to 0$,the cylinder becomes homogeneous ,the dispersion equation (18) and (22) can be reduced to

$$\tan(\sqrt{(\frac{c^2}{c_s^2}-1)}KH) = \frac{\mu_1}{\mu_0}\frac{\sqrt{(1-\frac{c^2}{c_1^2})}}{\sqrt{(\frac{c^2}{c_s^2}-1)}} \quad (23)$$

Where is the well known classical dispersion equation of Love wave [17].

4. Numerical Results and Discussions

In order to study the effect of various parameters on dispersion of torsional wave in a one-dimensional component of infinite length, we have taken the following date[18] to calculate the phase velocity from eqs.(24) for three diffrent values of elastic constants in Table 1.

Table 1. Values of Various E Constants

Curves	$\rho_0 / (kg \cdot m^{-3})$	$\mu_1 / (N \cdot m^{-2})$	$\rho_1 / (kg \cdot m^{-3})$	μ_0/μ_1	$(c_s/c_1)^2$
1	7450	7.5×10^{10}	4700	1.240	0.7823
2	7450	6.34×10^{10}	3364	1.4669	0.6624
3	7450	3.988×10^{10}	2649	2.3320	0.8292

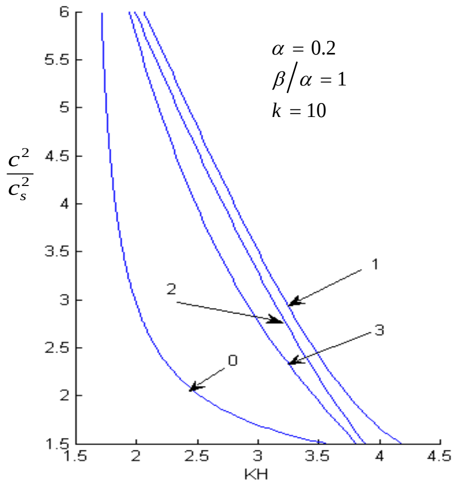

Figure 2. Dimensionless Phase Velocity Curves against Dimensionless Wave Number for case II when $\frac{\beta}{\alpha} = 2, \alpha = 0.2, K = 10$

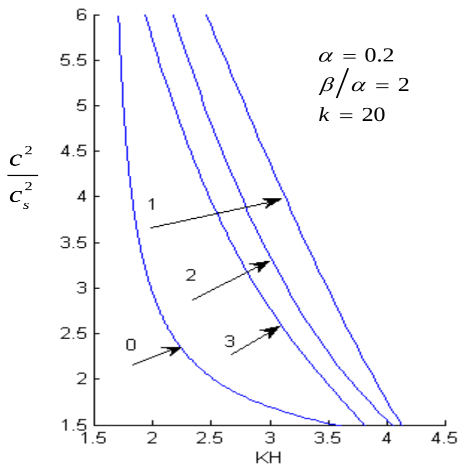

Figure 3. Dimensionless Phase Velocity Curves Against Dimensionless Wave Number for Case II When $\frac{\beta}{\alpha} = 2, \alpha = 0.2, K = 20$

Effect of linearly varing shear module and density on torsional wave in infinite one-dimensional component of Case I, Case II and Case III are discussed in the following way by means of Figure 2 and Figure 3.

(i) Curve 0 represents the classical case of Love wave in Figure 2 and Figure 3.with the increasing of value α ,dimensionless phase.

(ii) Curves 1,2,3 are plotted for Case II, respectively described the relations between dimensionless phase velocity $(c/c_s)^2$ and dimension wave number KH in three case of different constants when $\frac{\beta}{\alpha} = 2, \alpha = 0.2$, $K = 10$,and $\frac{\beta}{\alpha} = 1, \alpha = 0.2$ $K = 20$.

We can observed that

(i) Curve 0 is obviously steeper than the curves 1, 2, 3 in Figure 2 and Figure 3, the result is consistent with classical Love wave. Velocity $(c/c_s)^2$ is decreasing with the decreases in dimensionless wave number KH in all the cases.

(ii)The effect the linearly varying $\frac{\beta}{\alpha}$ from 1 to 2 and number K from 10 to 20, the phase velocity $(c/c_s)^2$ kept the same trend, but it is slower with the varying KH.

(iii)In the absence of α and β ,that is $\alpha = \beta = 0$,the dispersion of equation (11) and (24) changes to the classical equation (25) and hence the modelling of construction and the solution of the problem are valid and feasible in present paper.

5. Conclusion

In the paper, we studied the propagation of torsional waves in a transverse and layered one-dimensional component of infinite length. We proposed a new method that one-dimensional component was divided into multilayers to model and analyzed it, discussed two cases concerning torsional wave dispersion in the homogeneous and tranverse layered. Further, the governing equations and solutions are obtained using asymptotic linear expansion of the Whittaker's function. Finally numerical results and simulation gave the figures with the change of parameters density and shear module of the component, so the prsent research on the dispersion of torsinal wave in one-dimensional component of infinite length can be useful in modelling and analyzing the particle displacement, bearings, engineering mechanics and so on.

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Authors

WU Di