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Soliton Solutions for Nonlinear Evolution Equations with Symbolic Computation

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Abstract

The study of exact solutions of nonlinear evolution equations plays an important role in soliton theory. In this paper, we concentrate on proposing to a new Wronskian condition for the Levi equations by utilizing the Wronskian technique, which will lead to a way of constructing new double Wronskian representation of solutions for the Levi equations. Furthermore, some soliton solutions from the particular form of Wronskian determinant solutions are obtained with the aid of the symbolic computation system Maple.

Keywords*: nonlinear evolution equations, soliton, Wronskian, exact solutions.*

1. Introduction

The study of exact solutions of linear and nonlinear evolution equations plays an important role in theory and applications [1-16]. Exact solutions to nonlinear partial differential equations play an essential role in the nonlinear science, especially they may provide much physical information and help one to understand the mechanism that governs these physical models [17-23]. The theory is well developed for the case of linear equations depending on one variable and nonlinear integrable equations depending on two variables and includes, in particular, various versions of the commutation methods, algebro-geometric methods and Darboux transformations. There is a wide variety of approaches to nonlinear problems for constructing traveling wave solutions. Some of these approaches are inverse scattering method [24-25], Hirotas bilinear method [26], homogeneous balance method[27], Weierstrass function method, symmetry method, sine-cosine method, tanh-coth method , the Exp-function method, the Jacobi elliptic function method and so on[28-42].

It is well known that the Wronskian technique is considered as one of efficient direct approaches to deriving soliton solutions to nonlinear evolution equations possessing bilinear forms. It provides direct verifications of solutions to bilinear equations by taking the advantage that special structure of a Wronskian contributes simple forms of its derivatives. It has been widely applied to solition equations to derive their Wronskian solutions, including soliton, rational solutions, positon solutions, negaton solutions, complexiton solutions, and so on. The important procedure of the Wronskian technique is to choose a general Wronskian condition equation set that the Wronskian element vector satisfies.

In this paper, we apply the double Wronskian technique to study of exact solutions of nonlinear evolution equations. For illustration, we apply this method to the Levi equations and and successfully construct many soliton solutions with the aid of the symbolic computation system Maple.

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2. The Levi Equations

Consider the isospectral problem

$$\phi_{xx}+u\phi_x+v\phi=\lambda\phi \tag{1}$$

and

$$\phi_t=A\phi+B\phi_x \tag{2}$$

from (1) and (2), we get

$$\begin{aligned} u_t&=-2A_x-B_{xx}+uB_x+u_xB-\lambda B_x,\\ v_t&=-A_{xx}-uA_x+2vB_x+v_xB+\lambda A_x, \end{aligned} \tag{3}$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_t=\begin{pmatrix} -\partial+u+u_x\partial^{-1} & 2 \\ 2v+v_x\partial^{-1} & \partial+u \end{pmatrix}\begin{pmatrix} B_x \\ -A_x \end{pmatrix}+B_0\begin{pmatrix} u_x \\ v_x \end{pmatrix}-\lambda\begin{pmatrix} B_x \\ -A_x \end{pmatrix}, \tag{4}$$

$$\begin{pmatrix} B_x \\ -A_x \end{pmatrix}=\sum_{j=0}^{n}\begin{pmatrix} b_{j,x} \\ -a_{j,x} \end{pmatrix}\lambda^{n-j}, \tag{5}$$

taking $B_0=\lambda^{n+1}$, from (4) we can have

$$L\begin{pmatrix} b_{j,x} \\ -a_{j,x} \end{pmatrix}=\begin{pmatrix} b_{j+1,x} \\ -a_{j+1,x} \end{pmatrix},\begin{pmatrix} b_0 \\ -a_0 \end{pmatrix}=\begin{pmatrix} u \\ v \end{pmatrix}, \tag{6}$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_t=L^{n+1}\begin{pmatrix} u_x \\ v_x \end{pmatrix}, n=0,1,2,\cdots \tag{7}$$

where

$$L=\begin{pmatrix} -\partial+u+u_x\partial^{-1} & 2 \\ 2v+v_x\partial^{-1} & \partial+u \end{pmatrix}.$$

We can obtain the following nonlinear evolution equations from (7)

$$u_t=-u_{xx}+2uu_x+2v_x, \tag{8}$$

$$v_t=v_{xx}+2(uv)_x. \tag{9}$$

Eqs. (8) and (9) are the Levi equations.

3. The Bilinear Equations

Through the dependent variable transformations

$$u=-(\ln\frac{g}{f})_x,\quad v=(\ln f)_{xx}, \tag{10}$$

(8) and (9) can be transformed into the bilinear equations

$$(D_t-D_x^2)f\cdot g=0, \tag{11}$$

$$D_x^2 f\cdot f=2gh, \tag{12}$$

$$D_tD_x f\cdot f=2D_xh\cdot g, \tag{13}$$

where D is the well-known Hirota's bilinear operator defined by

$$D_t^m D_x^n a \cdot b = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n a(x,t) b(x',t') \Big|_{x=x', t=t',} m, n = 0, 1, \cdots$$

In the following, we use the abbreviated notion of Freeman and Nimmo for the Wronskian and its derivatives. As well known, a Wronskian is defined by

$$W(\phi_1, \phi_2, \cdots \phi_N) = \begin{vmatrix} \phi_1 & \phi_1^{(1)} \cdots & \phi_1^{(N-1)} \\ \vdots & \vdots & \vdots \\ \phi_N & \phi_N^{(1)} & \phi_N^{(N-1)} \end{vmatrix}$$

where

$$\phi_j^{(l)} = \partial^l \phi_j / \partial x^l.$$

It can be denoted by the following compact form

$$W(\phi) = \left|\phi, \phi^{(1)}, \cdots, \phi^{(N-1)}\right| = \left|0, 1, \cdots, N-1\right| = \left|N-1\right|$$

4. Exact Solutions Theorem

The Levi equations (8) and (9) have the general double Wronskian solutions

$$f = W^{N+1, M+1}(\phi; \varphi) = \left|N; M\right|, \tag{14}$$

$$g = 2W^{N+2, M}(\phi; \varphi) = 2\left|N+1; M-1\right|, \tag{15}$$

$$h = 2W^{N, M+2}(\phi; \varphi) = 2\left|N-1; M+1\right|, \tag{16}$$

under the matrix equations

$$\phi_x = A\phi, \quad \psi_x = -A\psi, \tag{17}$$

$$\phi_t = -2\phi_{xx,} \quad \psi_t = 2\psi_{xx,} \tag{18}$$

where

$$W^{N+1, M+1}(\phi; \varphi) = \left|\phi, \partial_x \phi, \cdots, \partial^{j-1}{}_x \phi; \varphi, \partial_x \varphi, \cdots, \partial^{j-1}{}_x \varphi\right| = \left|N; M\right|,$$

is double Wronskian and $A = (a_{ij})$ is a $(N+M+2) \times (N+M+2)$ arbitrary real matrix independent of x and t.

From (17) and (18), we can get the general solution

$$\phi = e^{-2A^2 t + Ax} C = \sum_{s=0}^{\infty} \left(\sum_{l=0}^{\left[\frac{s}{2}\right]} \frac{(-1)^l 2^l}{l!(s-2l)!} x^{s-2l} t^l \right) A^s C, \tag{19}$$

$$\psi = e^{2A^2 t - Ax} D = \sum_{s=0}^{\infty} \left(\sum_{l=0}^{\left[\frac{s}{2}\right]} \frac{(-1)^{s-2l} 2^l}{l!(s-2l)!} x^{s-2l} t^l \right) A^s D, \tag{20}$$

where $C=(c_1,c_2,\cdots,c_{N+M+2})^T, D=(d_1,d_2,\cdots,d_{N+M+2})^T$ are real constant vectors.

6. Soliton Solutions

Taking

$$A=\begin{pmatrix} k_1 & & & \\ & k_2 & & \\ & & \ddots & \\ & & & k_{M+N+2} \end{pmatrix},\ k_i \neq k_j (i\neq j), \tag{21}$$

we can obtain

$$\phi_j = c_j e^{-2k_j t - k_j x},$$
$$\varphi_j = d_j e^{2k_j t - k_j x},$$

then we can get soliton solutions of the the Levi equations (8) and (9).

For example, when $M=N=0$, we get

$$u=\frac{2(d_1k_1c_2e^{(k_2-k_1)(-2tk_2+x-2tk_1)}-d_2k_2c_1e^{(k_1-k_2)(-2tk_2+x-2tk_1)}}{d_2c_1e^{(k_1-k_2)(-2tk_2+x-2tk_1)}-d_1c_2e^{(k_2-k_1)(-2tk_2+x-2tk_1)}}, \tag{22}$$

$$v=\frac{4c_1c_2d_1d_2(k_1^2-2k_2k_1+k_2^2)}{\left(d_2c_1e^{(k_1-k_2)(-2tk_2+x-2tk_1)}-d_1c_2e^{(k_2-k_1)(-2tk_2+x-2tk_1)}\right)^2}. \tag{23}$$

When $M=0,\ N=1$, and $c_1=1,c_2=1,c_3=1,d_1=1,d_2=1,d_3=1,k_1=1,k_2=2,k_3=3$, we can obtain

$$\mathrm{u}=\frac{4\left(-6+e^{(20t-2x)}+e^{(-20t+2x)}-e^{(32t-4x)}-e^{(-32t+4x)}+3e^{(-12t+2x)}+3e^{(12t-2x)}\right)}{\left(-e^{8t}+2e^{(-12t+2x)}-e^{(-24t+4x)}\right)\left(-e^{-8t}+2e^{(12t-2x)}-e^{(24t-4x)}\right)}, \tag{24}$$

$$v=\frac{8e^{(28t-6x)}\left(-e^{8t}+2e^{(-12t+2x)}-e^{(-24t+4x)}\right)}{\left(-e^{-8t}+2e^{(12t-2x)}-e^{(24t-4x)}\right)^2}. \tag{25}$$

If taking $c_1=0,c_2=1,c_3=1,d_1=1,d_2=1,d_3=1,k_1=1,k_2=2,k_3=0$, we get

$$\mathrm{u}=-\frac{4e^{-3x+10t}}{e^{x-6t}+e^{-3x+10t}}, \tag{26}$$

$$v=-\frac{16e^{-2x+4t}}{\left(e^{x-6t}+e^{-3x+10t}\right)^2}. \tag{27}$$

And when $c_1=0,c_2=1,c_3=1,d_1=1,d_2=1,d_3=1,k_1=1,k_2=2,k_3=3$,

$$\mathrm{u}=-\frac{2\left(-3e^{8t}+4e^{(-12t+2x)}-e^{(-24t+4x)}\right)}{\left(-e^{8t}+2e^{(-12t+2x)}-e^{(-24t+4x)}\right)}\quad , \tag{28}$$

$$v=\frac{8e^{(-28t+6x)}\left(-e^{-8t}+2e^{(12t-2x)}-e^{(24t-4x)}\right)}{\left(-e^{8t}+2e^{(-12t+2x)}-e^{(-24t+4x)}\right)^2}\quad . \tag{29}$$

When $c_1=1, c_2=-1, c_3=1, d_1=1, d_2=0, d_3=1, k_1=1, k_2=0, k_3=3$,

$$\mathrm{u}=-\frac{6(e^{-2x+16t}-e^{2x-16t})}{e^{-2x+16t}-3e^{2x-16t}}\quad , \tag{30}$$

$$v=-\frac{48}{\left(e^{-2x+16t}-3e^{2x-16t}\right)^2}\quad . \tag{31}$$

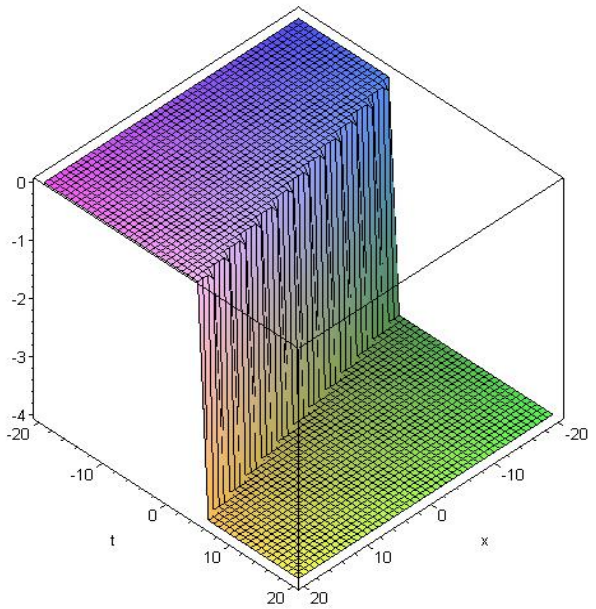

Figure 1. Soliton Solution for Expression (26)

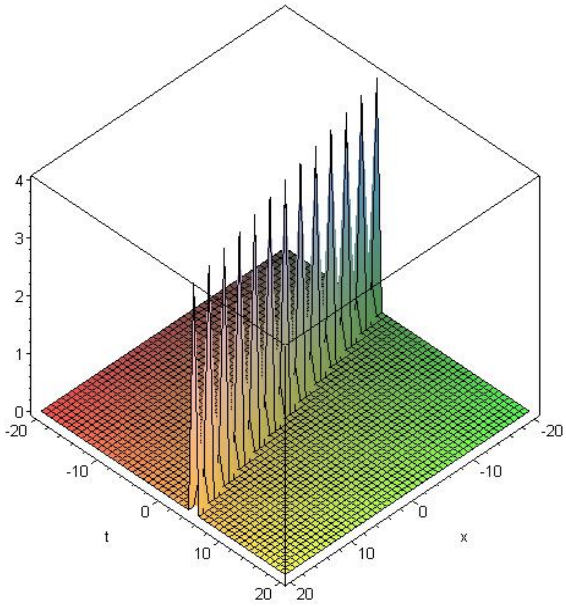

Figure 2. Soliton Solution for Expression (27)

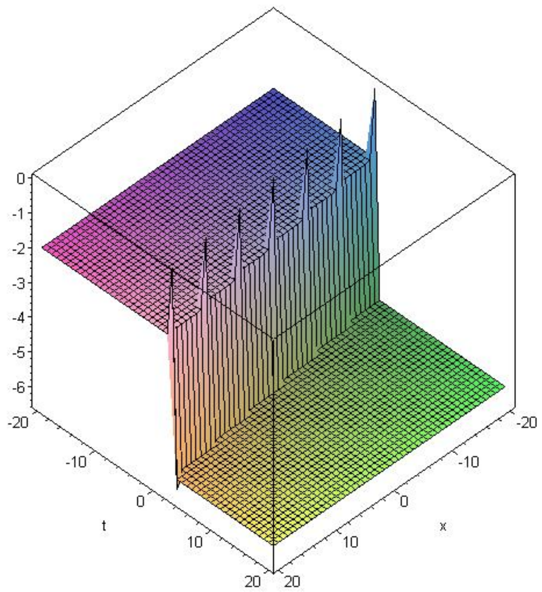

Figure 3. Soliton Solution for Expression (30)

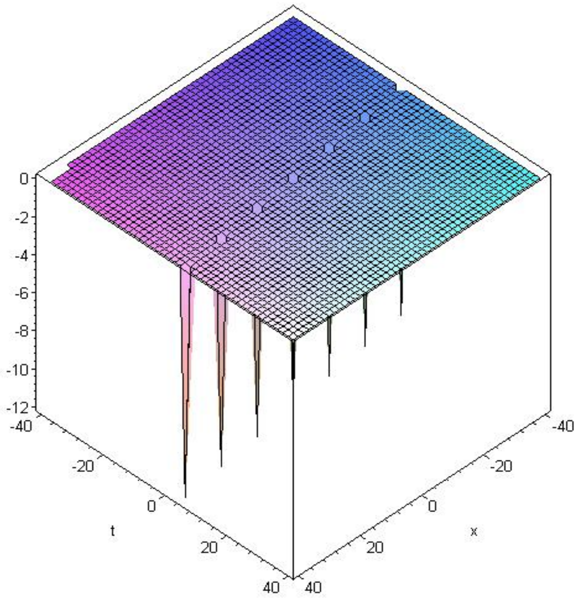

Figure 4. Soliton Solution for Expression (31)

7. Conclusion

In summary, the general double Wronskian method has been proposed and applied to construct exact solutions of the isospectral Levi equations. With the aid of Maple, we have obtained many soliton solutions. The general double Wronskian methodis more powerful in searching for exact solutions of nonlinear evolution equations.

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