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Alternating Iterative Algorithms for Split Equality Problem of Strictly Pseudononspreading Mapping

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Abstract

The purpose of this article is to prove the weak convergence theorems for solving split equality problem of strictly pseudononspreading mapping by introducing two alternating iterative algorithms. Furthermore, we apply our iterative algorithms to some convex and nonlinear problems. The results shown in this paper improve and extend the recent ones announced by many others.

Keywords*: Fixed point, Feasibility problem, CQ-algorithm; alternating algorithm, strictly nonspresding mapping.*

1. Introduction

Due to their extraordinary utility and broad applicability in many areas of applied mathematics (most notably, fully-discretized models of problems in image reconstruction from projections, in image processing, and in intensity-modulated radiation therapy), algorithms for solving convex feasibility problems continue to receive great attention; see for instance [1-9]. Recently, Moudafi cite [10] introduced a new convex feasibility problem (CFP). Let H_1, H_2, H_3 be real Hilbert spaces, let $C \subset H_1$, $Q \subset H_2$ be two nonempty closed convex sets, let $A: H_1 \to H_3$, B: $H_2 \to H_3$ be two bounded linear operators. The convex feasibility problem in [10] is to find

$$x \in C\ ,\ y \in Q \text{ such that } Ax = By \tag{1.1}$$

which allows asymmetric and partial relations between the variables x and y. The interest is to cover many situations, for instance in decomposition methods for PDEs, applications in game theory and in intensity-modulated radiation therapy (IMRT). In decision sciences, this allows to consider agents who interplay only via some components of their decision variables, for further details, the interested reader is referred to [11]. In IMRT, this amounts to envisage a weak coupling between the vector of doses absorbed in all voxels and that of the radiation intensity, for further details, the interested reader is referred to[11,12].

For solving the CFP (1.1), Moudafi [10] studied the fixed point formulation of the solutions of the CFP (1.1). Assuming that the CFP (1.1) is consistent (i.e., (1.1) has a solution), if (x, y) solves (1.1), then it solves the following fixed point equation system

$$\begin{cases} x = P_C(x - \gamma A^*(Ax - By)) \\ y = P_Q(y + \beta B^*(Ax - By)) \end{cases} \tag{1.2}$$

where $\gamma, \beta > 0$ are any positive constants. Moudafi [10] introduced the following alternating CQ algorithm

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$$\begin{cases} x_{k+1} = P_C(x_k - \gamma_k A^*(Ax_k - By_k)) \\ y_{k+1} = P_Q(y_k + \beta_k B^*(Ax_{k+1} - By_k)) \end{cases} \tag{1.3}$$

where $\gamma_k \in (\varepsilon, \min(\frac{1}{\lambda_A}, \frac{1}{\lambda_B}) - \varepsilon)$, λ_A and λ_B are the the spectral radiuses of A^*A and B^*B , respectively. Then he proved the weak convergence of the sequence (x_k, y_k) to a solution of (1.1) under some conditions.

In [10], Moudafi introduced the following problem

$$x \in F(U) \text{ , } y \in F(T) \text{ such that } Ax = By \tag{1.4}$$

and proposed the following alternating algorithm

$$\begin{cases} x_{k+1} = U(x_k - \gamma_k A^*(Ax_k - By_k)) \\ y_{k+1} = T(y_k + \beta_k B^*(Ax_{k+1} - By_k)) \end{cases} \tag{1.5}$$

for firmly quasi-nonexpansive operators U and T . Then he proved the weak convergence of the sequence (x_k, y_k) to a solution of (1.1) provided that the solution set $\Gamma := \{x \in F(U), y \in F(T); Ax = By\}$ is nonempty and some conditions on the sequence of positive parameters $\{\gamma_k\}$.

In this article, motivated by above results, we propose the following alternating Mann iterative algorithm for solving split equality problem

$$x \in \bigcap_{i=1}^{N} F(T_i), y \in \bigcap_{i=1}^{N} F(S_i) \text{ such that } Ax = By \tag{1.6}$$

where T_i is ρ_i -strictly pseudononspreading mapping and S_i is τ_i -strictly pseudononspreading mapping.

Algorithm 1.1 Let $x_0 \in H_1$, $y_0 \in H_2$ be arbitrary.

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k) \\ x_{k+1} = (1-\alpha_k)u_k + \alpha_k T_{n(\mathrm{mod}N)} u_k \\ v_{k+1} = y_k + \gamma_k B^*(Ax_{k+1} - By_k) \\ y_{k+1} = (1-\beta_k)v_{k+1} + \beta_k S_{n(\mathrm{mod}N)} v_{k+1} \end{cases}$$

The CQ algorithm is a special case of the K-M algorithm. We apply the K-M algorithm to solve (1.6) for strictly pseudononspreading mappings.

Algorithm 1.2 Let $x_0 \in H_1$, $y_0 \in H_2$ be arbitrary.

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k) \\ x_{k+1} = \alpha_k x_k + \beta_k u_k + t_k T_{n(\mathrm{mod}N)} u_k \\ v_{k+1} = y_k + \gamma_k B^*(Ax_{k+1} - By_k) \\ y_{k+1} = \alpha_k y_k + \beta_k v_{k+1} + t_k S_{n(\mathrm{mod}N)} v_{k+1} \end{cases}$$

2. Preliminaries

Throughout this paper, we denote by H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$, and denote by C be a nonempty closed convex subset of H . Let $T : H \to H$ be a mapping. A point $x \in H$ is said to be a fixed point of T

provided $x = Tx$. We use $F(T)$ to denote the fixed point set and use Γ stand for the solution set of the problem (1.6). We write $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\|, \forall y \in C.$$

Next, we review some definitions and basic results.

A mapping $T : H \to H$ is called firmly quasi-nonexpansive, if

$$\|Tx - q\|^2 \le \|x - q\|^2 + \|x - Tx\|^2, \forall (x,q) \in H \times F(T).$$

A mapping $T : H \to H$ is called quasi-nonexpansive, if

$$\|Tx - q\| \le \|x - q\|, \forall (x,q) \in H \times F(T).$$

Let K be a nonempty closed convex subset of a real Hilbert space H. A mapping $T : K \to K$ is called nonspresding, if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle, \forall x, y \in K.$$

A mapping $T : D(T) \subseteq H \to K$ is called is called k-strictly pseudononspresding if exists $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\,\|x - Tx - (y - Ty)\|^2 + 2\langle x - Tx, y - Ty\rangle,$$
$$\forall x, y \in D(T).$$

Remark 2.1 T is nonspresding mapping if and only if T is 0-strictly pseudononspresding mapping. If T is pseudononspresding mapping and the set of fixed point in nonempty, then T is quasi-nonexpansive mapping.

The so-called demiclosedness principle plays an important role in our argument.

A mapping $T : H \to H$ is called demi-closed at the origin if for any sequence $\{x_n\}$ which weakly converges to x, and if the sequence $\{x_n\}$ strongly converges to 0, then $Tx = 0$.

To establish our results, we need the following technical lemma.

Lemma 2.1 ([14]) If $x, y, z \in H$, then

(a) $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$.

(b) For any $\lambda \in [0,1]$

$$\|\lambda x + (1-\lambda) y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2.$$

(c) For $a,b,c \in [0,1]$ with $a + b + c = 1$,

$$\|ax + by + cz\|^2 = a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - ab\|x - y\|^2 - ac\|x - z\|^2 - bc\|y - z\|^2$$

3. Main Results

Now, we are in a position to prove our convergence result.

Theorem 3.1 Let H_1, H_2, H_3 be real Hilbert spaces. For $i = 1, 2, \cdots, N$, let T_i be a ρ_i-strictly pseudononspreading mapping and let S_i be a τ_i-strictly pseudononspreading mapping with nonempty fixed point set $F(T_i)$ and $F(S_i)$. Let $\{\gamma_k\}$ be a positive non-decreasing sequence such that $\gamma_k \in (\varepsilon, \min(\frac{1}{\lambda_A}, \frac{1}{\lambda_B}) - \varepsilon)$, where λ_A and λ_B are the spectral radiuses of A^*A and B^*B, respectively, and ε is small enough. If $T_i - I$, $S_i - I$ are demi-closed at origin, and the solution set Γ of (1.6) is nonempty, then the sequence $\{(x_k, y_k)\}$ generated by Algorithm 1.1 weakly converges to a solution (x^*, y^*) of (1.6), provided that $\alpha_k \subset (\mu, 1-\rho-\mu)$ and $\beta_k \subset (\delta, 1-\tau-\delta)$ for small enough $\mu, \delta > 0$ and $\rho = \max\{\rho_1, \rho_2, \cdots, \rho_N\} \in (0,1)$, $\tau = \max\{\tau_1, \tau_2, \cdots, \tau_N\} \in (0,1)$. Moreover, $\|Ax_k - By_k\| \to 0$, $\|x_k - x_{k+1}\| \to 0$ and $\|y_k - y_{k+1}\| \to 0$ as $k \to \infty$.

Proof for $(x, y) \in \Gamma$, from Algorithm 1.1, we have

$$\begin{aligned}\|u_k - x\|^2 &= \|x_k - \gamma_k A^*(Ax_k - By_k) - x\|^2 \\ &= \|x_k - x\|^2 - 2\gamma_k \langle x_k - x, A^*(Ax_k - By_k)\rangle + \gamma_k^2 \|A^*(Ax_k - By_k)\|^2.\end{aligned} \tag{3.1}$$

It follows from the definition of λ_A that

$$\begin{aligned}\gamma_k^2 \|A^*(Ax_k - By_k)\|^2 &= \gamma_k^2 \langle A^*(Ax_k - By_k), A^*(Ax_k - By_k)\rangle \\ &= \gamma_k^2 \langle Ax_k - By_k, AA^*(Ax_k - By_k)\rangle \\ &\le \lambda_A \gamma_k^2 \langle Ax_k - By_k, Ax_k - By_k\rangle \\ &= \lambda_A \gamma_k^2 \|Ax_k - By_k\|^2.\end{aligned} \tag{3.2}$$

Notice that

$$\begin{aligned}-2\langle x_k - x, A^*(Ax_k - By_k)\rangle &= -2\langle Ax_k - Ax, Ax_k - By_k\rangle \\ &= -\|Ax_k - Ax\|^2 - \|Ax_k - By_k\|^2 - \|By_k - Ax\|^2.\end{aligned} \tag{3.3}$$

Hence, substituting (3.2) and (3.3) into (3.1), we obtain

$$\begin{aligned}\|u_k - x\|^2 \le{}& \|x_k - x\|^2 - \gamma_k (1 - \lambda_A \gamma_k)\|Ax_k - By_k\|^2 \\ &- \gamma_k \|Ax_k - Ax\|^2 + \gamma_k \|By_k - Ax\|^2.\end{aligned} \tag{3.4}$$

Similarly, by Algorithm 1.1, we deduce

$$\begin{aligned}\|v_{k+1} - y\|^2 \le{}& \|y_k - y\|^2 - \gamma_k (1 - \lambda_B \gamma_k)\|Ax_{k+1} - By_k\|^2 \\ &- \gamma_k \|By_k - By\|^2 + \gamma_k \|By - Ax_{k+1}\|^2.\end{aligned} \tag{3.5}$$

Furthermore, from the fact that $Ax = By$ and the assumptions on $\{\gamma_k\}$, we have

$$\begin{aligned}\|u_k - x\|^2 + \|v_{k+1} - y\|^2 \le{}& \|x_k - x\|^2 + \|y_k - y\|^2 - \gamma_k \|Ax_k - Ax\|^2 + \gamma_{k+1}\|Ax_{k+1} - Ax\|^2 \\ &- \gamma_k (1 - \lambda_B \gamma_k)\|Ax_{k+1} - By_k\|^2 - \gamma_k (1 - \lambda_A \gamma_k)\|Ax_k - By_k\|^2.\end{aligned} \tag{3.6}$$

Next, we estimate $\|x_{k+1} - x\|$. Since T_i is ρ_i-strictly pseudononspreading mapping, one has

$$\begin{aligned}\left\langle u_k - x, u_k - T_{n(\bmod N)}u_k\right\rangle &= -\frac{1}{2}\left\|T_{n(\bmod N)}u_k - x\right\|^2 + \frac{1}{2}\left\|u_k - x\right\|^2 + \frac{1}{2}\left\|u_k - T_{n(\bmod N)}u_k\right\|^2 \\ &= -\frac{1}{2}\left\|T_{n(\bmod N)}u_k - x\right\|^2 + \frac{1-\rho}{2}\left\|u_k - T_{n(\bmod N)}u_k\right\|^2 \\ &\quad + \frac{1}{2}\left(\left\|u_k - x\right\|^2 + \rho\left\|u_k - T_{n(\bmod N)}u_k\right\|^2\right) \\ &\geq \frac{1-\rho}{2}\left\|u_k - T_{n(\bmod N)}u_k\right\|^2 .\end{aligned}$$

(3.7)

It follows from Algorithm 1.1 that

$$\begin{aligned}\left\|x_{k+1} - x\right\|^2 &= \left\|(1-\alpha_k)u_k - \alpha_k T_{n(\bmod N)}u_k - x\right\|^2 \\ &= \left\|u_k - x - \alpha_k\left(u_k - T_{n(\bmod N)}u_k\right)\right\|^2 \\ &= \left\|u_k - x\right\|^2 + \alpha_k^2\left\|u_k - T_{n(\bmod N)}u_k\right\|^2 - 2\alpha_k\left\langle u_k - x, u_k - T_{n(\bmod N)}u_k\right\rangle \\ &\leq \left\|u_k - x\right\|^2 - \alpha_k\left(1-\rho-\alpha_k\right)\left\|u_k - T_{n(\bmod N)}u_k\right\|^2 .\end{aligned} \tag{3.8}$$

Similarly, we have

$$\left\|y_{k+1} - x\right\|^2 \leq \left\|v_{k+1} - x\right\|^2 - \beta_k\left(1-\tau-\beta_k\right)\left\|v_{k+1} - S_{n(\bmod N)}v_{k+1}\right\|^2 . \tag{3.9}$$

Thus, (3.8) and (3.9) lead to

$$\begin{aligned}\left\|x_{k+1} - x\right\|^2 + \left\|y_{k+1} - x\right\|^2 \leq{}& \left\|x_k - x\right\|^2 + \left\|y_k - y\right\|^2 - \gamma_k\left\|Ax_k - Ax\right\|^2 + \gamma_{k+1}\left\|Ax_{k+1} - Ax\right\|^2 \\ &- \gamma_k\left(1-\lambda_B\gamma_k\right)\left\|Ax_{k+1} - By_k\right\|^2 - \gamma_k\left(1-\lambda_A\gamma_k\right)\left\|Ax_k - By_k\right\|^2 \\ &- \alpha_k\left(1-\rho-\alpha_k\right)\left\|u_k - T_{n(\bmod N)}u_k\right\|^2 \\ &- \beta_k\left(1-\tau-\beta_k\right)\left\|v_{k+1} - S_{n(\bmod N)}v_{k+1}\right\|^2 .\end{aligned} \tag{3.10}$$

Now, setting $\theta_k(x,y) = \left\|x_k - x\right\|^2 + \left\|y_k - y\right\|^2 - \gamma_k\left\|Ax_k - Ax\right\|^2$, we obtain the following inequality

$$\begin{aligned}\theta_{k+1}(x,y) \leq{}& \theta_k(x,y) - \gamma_k\left(1-\lambda_B\gamma_k\right)\left\|Ax_{k+1} - By_k\right\|^2 - \gamma_k\left(1-\lambda_A\gamma_k\right)\left\|Ax_k - By_k\right\|^2 \\ &- \alpha_k\left(1-\rho-\alpha_k\right)\left\|u_k - T_{n(\bmod N)}u_k\right\|^2 - \beta_k\left(1-\tau-\beta_k\right)\left\|v_{k+1} - S_{n(\bmod N)}v_{k+1}\right\|^2 .\end{aligned} \tag{3.11}$$

On the other hand, note that

$$\gamma_k\left\|Ax_k - Ax\right\|^2 = \gamma_k\left\langle x_k - x, A^*\left(Ax_k - Ax\right)\right\rangle \leq \gamma_k\lambda_A\left\|x_k - x\right\|^2,$$

which implies

$$\theta_k(x,y) \geq \left(1-\gamma_k\lambda_A\right)\left\|x_k - x\right\|^2 + \left\|y_k - y\right\|^2 \geq 0 . \tag{3.12}$$

The sequence $\theta_k(x,y)$ being decreasing and lower bounded by 0, consequently converges to some finite limit, says $\theta(x,y)$. And from (3.11), we obtain

$$\theta_{k+1}(x,y) \leq \theta_k(x,y) - \gamma_k\left(1-\lambda_A\gamma_k\right)\left\|Ax_k - By_k\right\|^2,$$

and hence

$$\lim_{k\to\infty}\|Ax_k - By_k\| = 0. \tag{3.13}$$

By the conditions on $\{\gamma_k\}$, $\{\alpha_k\}$ and $\{\beta_k\}$, one has

$$\lim_{k\to\infty}\|Ax_{k+1} - By_k\| = \lim_{k\to\infty}\|u_k - T_{n(\mathrm{mod}N)}u_k\| = \lim_{k\to\infty}\|v_{k+1} - S_{n(\mathrm{mod}N)}v_{k+1}\| = 0.$$

Since

$$\|u_k - x_k\| = \gamma_k \left\|A^*\left(Ax_k - By_k\right)\right\|,$$

we obtain

$$\lim_{k\to\infty}\|u_k - x_k\| = 0, \tag{3.14}$$

which means

$$\lim_{k\to\infty}\|T_{n(\mathrm{mod}N)}u_k - x_k\| = 0. \tag{3.15}$$

Furthermore, the equation $\|v_{k+1} - y_k\| = \gamma_k \left\|B^*\left(Ax_{k+1} - By_k\right)\right\|$ leads to

.
$$\lim_{k\to\infty}\|v_{k+1} - y_k\| = 0, \tag{3.16}$$

Let us now prove that $\{x_k\}$ and $\{y_k\}$ are asymptotically regular. Indeed, since

$$\|x_{k+1} - x_k\| \le \left(1-\alpha_k\right)\|u_k - x_k\| + \alpha_k \left\|T_{n(\mathrm{mod}N)}u_k - u_k\right\|,$$

from (3.14) and (3.15), we show that $\{x_k\}$ is asymptotically regular, namely $\lim_{k\to\infty}\|x_{k+1} - x_k\| = 0$. Similarly $\{y_k\}$ is asymptotically regular, too.

It follows from (3.12) and the assumption on $\{\gamma_k\}$ that

$$\theta_k\left(x, y\right) \ge \varepsilon\lambda_A \|x_k - x\|^2 + \|y_k - y\|^2, \tag{3.17}$$

which implies that both sequences $\{x_k\}$ and $\{y_k\}$ are bounded thanks to the fact that $\theta_k\left(x, y\right)$ converges to a finite limit.

Let x^* and y^* be respectively weak cluster points of the sequences $\{x_k\}$ and $\{y_k\}$, then there exist two subsequences of $\{x_k\}$ and $\{y_k\}$ (again labeled $\{x_k\}$ and $\{y_k\}$ which converge weakly to x^* and y^*). Next, we will show that $\left(x^*, y^*\right) \in \Gamma$. From (3.14), there exists a subsequence $\{u_{k_i}\} \subset \{u_k\}$ such that $\{u_{k_i}\} \to x^*$, hence for any positive integer $j = 1, 2, \cdots, N$, there exists a subsequence $\{k_i\left(j\right)\} \subset \{k_i\}$ with $k_i\left(j\right)\left(\mathrm{mod}N\right) = j$ such that $\{u_{k_i(j)}\} \to x^*$. Again, by (3.15), we know that $\|u_{iN+j} - T_j u_{iN+j}\| \to 0$, as $n \to \infty$. Thus, we obtain $\left\|u_{k_i(j)} - T_j u_{k_i(j)}\right\| \to 0$, as $k_{i(j)} \to \infty$. Since $T_j - I$ is demi-closed at zero, it follows that $x^* \in F\left(T_j\right)$. Similarly, we have $y^* \in F\left(S_j\right)$. Furthermore, the weak convergence of $Ax_k - By_k$ to $Ax^* - By^*$ and the lower semi-continuity of the squared norm imply

$$\left\|Ax^* - By^*\right\| \le \liminf_{k\to\infty} \|Ax_k - By_k\| = 0,$$

Hence $\left(x^*, y^*\right) \in \Gamma$.

Next, we will show the uniqueness of the weak cluster points of $\{x_k\}$ and $\{y_k\}$. Indeed, let $\bar{x}$, $\bar{y}$ be other weak cluster points of $\{x_k\}$ and $\{y_k\}$, respectively. From the definition of $\theta_k(x,y)$, we have

$$\begin{aligned}\theta_k(x^*,y^*) &= \|x_k - x^*\|^2 + \|y_k - y^*\|^2 - \gamma_k \|Ax_k - Ax^*\|^2 \\ &= \|x_k - \bar{x}\|^2 + \|\bar{x} - x^*\|^2 + 2\langle x_k - \bar{x}, \bar{x} - x^*\rangle \\ &+ \|y_k - \bar{y}\|^2 + \|\bar{y} - y^*\|^2 + 2\langle y_k - \bar{y}, \bar{y} - y^*\rangle \\ &- \gamma_k\left(\|Ax_k - A\bar{x}\|^2 + \|A\bar{x} - Ax^*\|^2 - 2\langle Ax_k - A\bar{x}, A\bar{x} - Ax^*\rangle\right) \\ &= \theta_k(\bar{x},\bar{y}) + \|\bar{x} - x^*\|^2 + \|\bar{y} - y^*\|^2 - \gamma_k\|A\bar{x}_k - Ax^*\|^2 \\ &+ 2\langle x_k - \bar{x}, \bar{x} - x^*\rangle + 2\langle y_k - \bar{y}, \bar{y} - y^*\rangle - 2\gamma_k\langle Ax_k - A\bar{x}, A\bar{x} - Ax^*\rangle.\end{aligned} \tag{3.18}$$

Without loss of generality, we may assume that $x_k \to \bar{x}$, $y_k \to \bar{x}$ and $\gamma_k \to \gamma^*$, from the boundedness of the sequence $\{\gamma_k\}$, we have

$$\theta(x^*,y^*) = \theta(\bar{x},\bar{y}) + \|\bar{x} - x^*\|^2 + \|\bar{y} - y^*\|^2 - \gamma^*\|A\bar{x} - Ax^*\|^2. \tag{3.19}$$

Reversing the role of (x^*,y^*) and $(\bar{x},\bar{y})$, we obtain

$$\theta(\bar{x},\bar{y}) = \theta(x^*,y^*) + \|x^* - \bar{x}\|^2 + \|y^* - \bar{y}\|^2 - \gamma^*\|Ax^* - A\bar{x}\|^2. \tag{3.20}$$

Adding the two last equalities, one has

$$\varepsilon\lambda_A\|x^* - \bar{x}\| + \|y^* - \bar{y}\| \le 0,$$

which means $x^* = \bar{x}$ and $y^* = \bar{y}$. Hence, the sequence $\{(x_k, y_k)\}$ weakly converges to a solution of problem (1.1), which completes the proof.

Theorem 3.2 Let H_1, H_2, H_3 be real Hilbert spaces. For $i = 1, 2, \cdots N$, let T_i be a ρ_i-strictly pseudononspreading mapping and let S_i be a τ_i-strictly pseudononspreading mapping with nonempty fixed point set $F(T_i)$ and $F(S_i)$. Let $\{\gamma_k\}$ be a positive non-decreasing sequence such that $\gamma_k \in (\varepsilon, \min(\frac{1}{\lambda_A}, \frac{1}{\lambda_B}) - \varepsilon)$, where λ_A and λ_B are the spectral radiuses of A^*A and B^*B, respectively, and ε is small enough. If $T_i - I$, $S_i - I$ are demiclosed at origin, and the solution set Γ of (1.6) is nonempty, then the sequence $\{(x_k, y_k)\}$ generated by Algorithm 1.2 weakly converges to a solution (x^*, y^*) of (1.6), provided that $\{\alpha_k\}$ is a non-increasing sequence such that $\alpha_k \subset (\mu, 1 - \xi - \mu)$ for small enough μ, where $\xi = \max\{\rho, \tau\}$, $\rho = \max\{\rho_1, \rho_2, \cdots, \rho_N\} \in (0,1)$, $\tau = \max\{\tau_1, \tau_2, \cdots, \tau_N\} \in (0,1)$. Moreover, $\|Ax_k - By_k\| \to 0$, $\|x_k - x_{k+1}\| \to 0$ and $\|y_k - y_{k+1}\| \to 0$ as $k \to \infty$.

Proof For $(x, y) \in \Gamma$, repeating the proof of Theorem 3.1, we obtain (3.10) is true.

It follows from lemma 2.1 that

$$\begin{aligned}\|x_{n+1}-x\|^2 &= \|\alpha_k x_k+\beta_k u_k+t_k T_{n(\mathrm{mod}N)}u_k-x\|^2\\ &= \|\alpha_k(x_k-x)+\beta_k(u_k-x)+t_k T_{n(\mathrm{mod}N)}u_k-x\|^2\\ &\le \alpha_k\|x_k-x\|^2+\beta_k\|u_k-x\|^2+t_k\|T_{n(\mathrm{mod}N)}u_k-x\|^2-\alpha_k\beta_k\|u_k-x_k\|^2\\ &-\alpha_k t_k\|T_{n(\mathrm{mod}N)}u_k-x_k\|^2-\beta_k t_k\|T_{n(\mathrm{mod}N)}u_k-u_k\|^2\\ &\le \alpha_k\|x_k-x\|^2+\beta_k\|u_k-x\|^2+t_k\|u_k-x\|^2-\alpha_k\beta_k\|u_k-x_k\|^2\\ &-\alpha_k t_k\|T_{n(\mathrm{mod}N)}u_k-x_k\|^2-\beta_k t_k\|T_{n(\mathrm{mod}N)}u_k-u_k\|^2\\ &= \alpha_k\|x_k-x\|^2+(1-\alpha_k)\|u_k-x\|^2-\alpha_k\beta_k\|u_k-x_k\|^2\\ &-\alpha_k t_k\|T_{n(\mathrm{mod}N)}u_k-x_k\|^2-\beta_k t_k\|T_{n(\mathrm{mod}N)}u_k-u_k\|^2\end{aligned} \tag{3.21}$$

Similarly, we have

$$\begin{aligned}\|y_{n+1}-y\|^2 \le \alpha_k\|y_k-y\|^2+(1-\alpha_k)\|v_{k+1}-y\|^2-\alpha_k\beta_k\|y_k-v_{k+1}\|^2\\ -\alpha_k t_k\|S_{n(\mathrm{mod}N)}v_{k+1}-y_k\|^2-\beta_k t_k\|S_{n(\mathrm{mod}N)}v_{k+1}-v_{k+1}\|^2\end{aligned} \tag{3.22}$$

Adding the two last inequalities, one has

$$\begin{aligned}\|x_{n+1}-x\|^2+\|y_{n+1}-y\|^2 \le \alpha_k\left(\|x_k-x\|^2+\|y_k-y\|^2\right)+(1-\alpha_k)\left(\|u_k-x\|^2+\|v_{k+1}-y\|^2\right)\\ -\alpha_k\beta_k\left(\|u_k-x_k\|^2+\|y_k-v_{k+1}\|^2\right)\\ -\alpha_k t_k\left(\|T_{n(\mathrm{mod}N)}u_k-x_k\|^2+\|S_{n(\mathrm{mod}N)}v_{k+1}-y_k\|^2\right)\\ -\beta_k t_k\left(\|T_{n(\mathrm{mod}N)}u_k-u_k\|^2+\|S_{n(\mathrm{mod}N)}v_{k+1}-v_{k+1}\|^2\right).\end{aligned} \tag{3.23}$$

It follows from (3.6) that

$$\begin{aligned}\|x_{n+1}-x\|^2+\|y_{n+1}-y\|^2 \le \|x_k-x\|^2+\|y_k-y\|^2-\gamma_k(1-\alpha_k)\|Ax_k-Ax\|^2\\ +\gamma_{k+1}(1-\alpha_k)\|Ax_{k+1}-Ax\|^2-\gamma_k(1-\alpha_k)(1-\lambda_B\gamma_k)\|Ax_{k+1}-By_k\|^2\\ -\gamma_k(1-\alpha_k)(1-\lambda_A\gamma_k)\|Ax_k-By_k\|^2\\ -\alpha_k\beta_k\left(\|u_k-x_k\|^2+\|y_k-v_{k+1}\|^2\right)\\ -\alpha_k t_k\left(\|T_{n(\mathrm{mod}N)}u_k-x_k\|^2+\|S_{n(\mathrm{mod}N)}v_{k+1}-y_k\|^2\right)\\ -\beta_k t_k\left(\|T_{n(\mathrm{mod}N)}u_k-u_k\|^2+\|S_{n(\mathrm{mod}N)}v_{k+1}-v_{k+1}\|^2\right).\end{aligned} \tag{3.24}$$

Now, setting $\theta_k(x,y)=\|x_k-x\|^2+\|y_k-y\|^2-\gamma_k(1-\alpha_k)\|Ax_k-Ax\|^2$, we obtain the following inequality

$$
\begin{aligned}
\theta_{k+1}(x,y) \le \theta_k(x,y) &- \gamma_k(1-\alpha_k)(1-\lambda_B\gamma_k)\|Ax_{k+1}-By_k\|^2 \\
&- \gamma_k(1-\alpha_k)(1-\lambda_A\gamma_k)\|Ax_k-By_k\|^2 \\
&- \alpha_k\beta_k\left(\|u_k-x_k\|^2+\|y_k-v_{k+1}\|^2\right) \\
&- \alpha_k t_k\left(\|T_{n(\mathrm{mod}N)}u_k-x_k\|^2+\|S_{n(\mathrm{mod}N)}v_{k+1}-y_k\|^2\right) \\
&- \beta_k t_k\left(\|T_{n(\mathrm{mod}N)}u_k-u_k\|^2+\|S_{n(\mathrm{mod}N)}v_{k+1}-v_{k+1}\|^2\right).
\end{aligned} \tag{3.25}
$$

Following the lines of the proof of Theorem 3.1, we deduce that the Sequence $\{\theta_k(x,y)\}$ converges to some finite limit, say $\theta(x,y)$. Furthermore, we obtain

$$
\begin{aligned}
\lim_{k\to\infty}\|Ax_{k+1}-By_k\| = \lim_{k\to\infty}\|Ax_k-By_k\| &= \lim_{k\to\infty}\|u_k-x_k\| = \lim_{k\to\infty}\|y_k-v_{k+1}\| \\
&= \lim_{k\to\infty}\|T_{n(\mathrm{mod}N)}u_k-x_k\|^2 = \lim_{k\to\infty}\|S_{n(\mathrm{mod}N)}v_{k+1}-y_k\| \\
&= \lim_{k\to\infty}\|T_{n(\mathrm{mod}N)}u_k-u_k\| = \lim_{k\to\infty}\|S_{n(\mathrm{mod}N)}v_{k+1}-v_{k+1}\| = 0.
\end{aligned} \tag{3.26}
$$

Notice that

$$\|x_{k+1}-x_k\| = \beta_k\|u_k-x_k\| = t_k\|T_{n(\mathrm{mod}N)}u_k-x_k\|.$$

Thus, we have

$$\lim_{k\to\infty}\|x_{k+1}-x_k\| = 0.$$

which means that $\{x_k\}$ is asymptotically regular. Similarly, we obtain $\{y_k\}$ is asymptotically regular, too.

The rest of the proof is analogous to Theorem 3.1.

4. Applications

4.1. Convex Feasibility Problem

We now pay our attention to applying our alterative iterative algorithms to some convex and nonlinear analysis notions, see, for example, [15].

For $N=1$, if T and S are nonspresding mappings and and the set of fixed point in nonempty, then T and S are quasi-nonexpansive mappings. We obtain the following alterative iterative algorithms for convex feasibility problem (1.4).

Algorithm 4.1 Let, $x_0 \in H_1, y_0 \in H_2$ be arbitrary.

$$
\begin{cases}
u_k = x_k - \gamma_k A^*(Ax_k - By_k) \\
x_{k+1} = (1-\alpha_k)u_k + \alpha_k T u_k \\
v_{k+1} = y_k + \beta_k B^*(Ax_{k+1} - By_k) \\
y_{k+1} = (1-\beta_k)v_{k+1} + \beta_k S v_{k+1}
\end{cases}
$$

Algorithm 4.2. Let, $x_0 \in H_1, y_0 \in H_2$ be arbitrary.

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k) \\ x_{k+1} = \alpha_k x_k + \beta_k u_k + t_k T u_k \\ v_{k+1} = y_k + \beta_k B^*(Ax_{k+1} - By_k) \\ y_{k+1} = \alpha_k y_k + \beta_k v_{k+1} + t_k S v_{k+1} \end{cases}$$

Furthermore, if $T_{n(\mathrm{mod}N)} = P_C$ and $S_{n(\mathrm{mod}N)} = P_Q$, then we obtain the following alterative iterative algorithms for convex feasibility problem (1.1).

Algorithm 4.3. Let $x_0 \in H_1, y_0 \in H_2$ be arbitrary.

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k) \\ x_{k+1} = (1-\alpha_k)u_k + \alpha_k P_C u_k \\ v_{k+1} = y_k + \beta_k B^*(Ax_{k+1} - By_k) \\ y_{k+1} = (1-\beta_k)v_{k+1} + \beta_k P_Q v_{k+1} \end{cases}$$

Algorithm 4.4. Let $x_0 \in H_1, y_0 \in H_2$ be arbitrary.

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k) \\ x_{k+1} = \alpha_k x_k + \beta_k u_k + t_k P_C u_k \\ v_{k+1} = y_k + \beta_k B^*(Ax_{k+1} - By_k) \\ y_{k+1} = \alpha_k y_k + \beta_k v_{k+1} + t_k P_Q v_{k+1} \end{cases}$$

4.2. Variational Problems via Resolvent Mappings

Given a maximal monotone operator $M : H_1 \to 2^{H_1}$, it is well known that its associated resolvent mapping, $J_\mu^M = \left(I + \mu M\right)^{-1}$, is quasi-nonexpansive and $0 \in M\left(x\right) \Leftrightarrow x = J_\mu^M\left(x\right)$ which implies that zeroes of M are exactly fixed-points of its resolvent mapping. If $T_{n(\mathrm{mod}N)} = J_\mu^M$ and $S_{n(\mathrm{mod}N)} = J_\nu^N$, where $N : H_2 \to 2^{H_2}$ is another maximal monotone operator, the problem under consideration is nothing but find $x^* \in M^{-1}\left(0\right)$, $y^* \in N^{-1}\left(0\right)$ such that $Ax^* = Bx^*$, and the algorithms are applied the following form.

Algorithm 4.5. Let $x_0 \in H_1, y_0 \in H_2$ be arbitrary.

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k) \\ x_{k+1} = (1-\alpha_k)u_k + \alpha_k J_\mu^M u_k \\ v_{k+1} = y_k + \beta_k B^*(Ax_{k+1} - By_k) \\ y_{k+1} = (1-\beta_k)v_{k+1} + \beta_k J_\nu^N v_{k+1} \end{cases}$$

Algorithm 4.6. Let $x_0 \in H_1, y_0 \in H_2$ be arbitrary.

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k) \\ x_{k+1} = \alpha_k x_k + \beta_k u_k + t_k J_\mu^M u_k \\ v_{k+1} = y_k + \beta_k B^*(Ax_{k+1} - By_k) \\ y_{k+1} = \alpha_k y_k + \beta_k v_{k+1} + t_k J_v^N v_{k+1} \end{cases}$$

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