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Constrained State Estimation via Projection based Optimized Parameters UKF

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Abstract

The unscented Kalman filter (UKF) has become a popular method for nonlinear state estimation during the last decade. However, the conventional UKF may not be suitable for real-world applications with state constrains that stem from physical definitions, physical laws or model restrictions. A UKF based method with optimized parameters was proposed in this paper to handle state constraints via the projection of sigma points. In the proposed method, the generated sigma points that violate the state constraints were projected onto the constraint boundary first. The three free parameters of the UKF, i.e., a ,β ,κ , were then optimized using a Gaussian process optimization (GPO) method. Simulations indicate that the proposed optimized UKF algorithm with the projection of sigma points can handle constrained state estimation problem effectively and efficiently.

Keywords*: Constrained nonlinear state estimation, unscented Kalman filter, sigma points projection, parameters learning, Gaussian process optimization*

1. Introduction

State estimation of dynamic systems in wide range applications can be characterized by the following state space formulation

$$x_{k+1} = f(x_k, u_k) + \varepsilon_k \tag{1a}$$

$$y_k = h(x_k, u_k) + v_k \tag{1b}$$

where x is the state vector, y is the measurement, u is the exterior input, ε is the process noise with a covariance of R, and v is the measurement noise with a covariance of Q. The most popular method for solving linear state estimation problems is the Kalman filter (KF), while the most popular methods for solving nonlinear state estimation problems are the extended Kalman filter (EKF) [1], the unscented Kalman filter (UKF) [2] and the particle filter (PF) [3], *etc*. Recently, new extensions of the Kalman filter such as the cubature KF [4], the complex UKF [5], and the truncated UKF [6], the quadrature KF [20, 21] were proposed and studied to improve the state estimation accuracy.

In practical applications, nonlinear processes with constraints are commonly encountered. Some of these constraints stem from physical definitions, e.g., the battery state of charge (the ratio of the remaining amount of capacity to the nominal capacity) should be in a range of 0% to 100% [7]. Other constraints may arise from physical laws. The mass of constituents in a sealed chemical reactor, for example, must remain constant throughout the reaction process [8]. These constraints are usually in the form of algebraic equality or inequality relationships. Incorporation of such constraints into state estimation is no doubt necessary and will be useful for improving estimation performance.

A number of approaches has been developed to handle these equality and/or inequality constraints. For equality constrained state estimation, the measurement-augmentation KF was one of the most popular techniques. Besides that, the

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projection based methods such as estimation projection, system projection and gain projection were also widely used [9]. A two-step projection algorithm for handling nonlinear equality constraints was given in [10]. For inequality constrained state estimation, moving horizon estimators (MHE) [11] and constrained Bayesian estimators [12-14] were widely used. In these methods, the constraints were often embedded into a nonlinear constrained optimization problem and then solved with optimization methods. The optimization was done in each time step and it is generally time-consuming.

In this paper, a parameters optimized UKF algorithm was proposed for constrained nonlinear state estimation. The constraints were handled in the following manner: In the time update step of the UKF algorithm, those sigma points violating the constraints were projected onto the constraint surface first. Then, the weighted sum of the new set of sigma points was used as the state prediction. The main difference between this method and the state of the art is that the parameterized UKF is treated as a Gaussian process (GP) [15] and its three free parameters, *i.e.*, α,β,κ, are determined using a GPO method [19]. The constraints were handled using the same projection technique that is used in the parameter learning (or parameter optimization) process. A flowchart of the proposed method is shown in Figure 1.

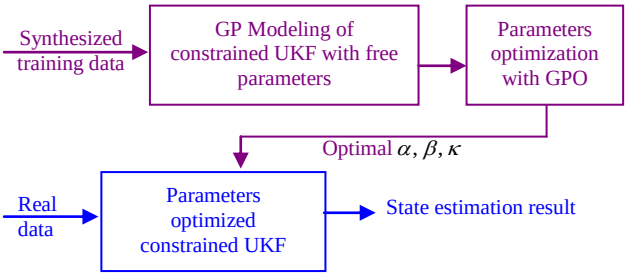


Figure 1. Flowchart of the Proposed Parameters Learnt Constrained UKF

The remainder of this paper is organized as follows: Section II gives a brief review of the generic UKF. Section III shows three methods for constraint handling in UKF. Section IV describes the parameters learning algorithm for projection based UKF with GPO. Section V provides the simulation results and the summary is given in Section V.

2. Brief Review of the Generic UKF

In the generic UKF algorithm, the distribution of the state x_k is specified using a minimal set of $2n+1$ carefully chosen sigma points $\left\{X_k^i (i=0,1,2,...,2n)\right\}$:

$$\begin{aligned} X_k^0 &:= \hat{x}_{k-1} \\ X_k^i &:= \hat{x}_{k-1} + \left(\sqrt{(n+\lambda)P_{k-1}}\right)_i, (i=1,2,...,n) \\ X_k^i &:= \hat{x}_{k-1} - \left(\sqrt{(n+\lambda)P_{k-1}}\right)_i, (i=n+1,n+2,...,2n) \end{aligned} \tag{3}$$

where n is the dimension of the state, $\lambda := \alpha^2(n+\kappa)-n$, $\hat{x}_{k-1}$ is the state estimation of x_{k-1}, P_{k-1} is the state estimation covariance, $\sqrt{}_i$ refers to i-th row of the Cholesky factorization, and the sigma points have weights assigned by

$$w_m^0 = \frac{\lambda}{n+\lambda}, w_c^0 = \frac{\lambda}{n+\lambda} + (1-\alpha^2+\beta),$$
$$w_m^i = w_c^i = \frac{\lambda}{2(n+\lambda)}, \quad i = 1,2,...,2n \tag{4}$$

where w_m is used to reconstruct the predicted mean, w_c is used to reconstruct the covariance, and $\{\alpha, \beta, \kappa\} \in R^+$ are three free parameters of the UKF algorithm. With these definitions, the steps of the generic UKF can be summarized as follows:

- Initialization: $\hat{x}_0 = E[x_0]$, $P_0 = E[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T]$
- For $k = 1,2,...$, do the following steps:

(1) Calculate the $2n+1$ sigma points X_k^i according to (3).

(2) Time update:

Propagated states $X_{k|k-1}^i$: $X_{k|k-1}^i = f(X_{k-1}^i, u_k)$

Predicted state x_k^-: $x_k^- = \sum_{i=0}^{2n} w_m^i X_{k|k-1}^i$

and its covariance P_k^-: $P_k^- = \sum_{i=0}^{2n} w_c^i \left(X_{k|k-1}^i - x_k^- \right)\left(X_{k|k-1}^i - x_k^- \right)^{\mathrm{T}}$

(3) Measurement update:

Propagated measurement $Y_{k|k-1}$: $Y_{k|k-1}^i = h(X_{k|k-1}^i, u_k)$

Predicted measurement y_k^-: $y_k^- = \sum_{i=0}^{2n} w_m^i Y_{k|k-1}^i$

and its covariance P_k^{y-}: $P_k^{y-} = \sum_{i=0}^{2n} w_c^i (Y_{k|k-1}^i - y_k^-)(Y_{k|k-1}^i - y_k^-)^T$

Cross-covariance P_k^{xy}: $P_k^{xy} = \sum_{i=0}^{2n} w_c^i (X_{i,k|k-1} - x_k^-)(Y_{i,k|k-1} - y_k^-)$

Kalman gain K_k: $K_k = P_k^{xy}(P_k^{y-})^{-1}$

Estimated state $\hat{x}_k$: $\hat{x}_k = x_k^- + K_k(y_k - y_k^-)$

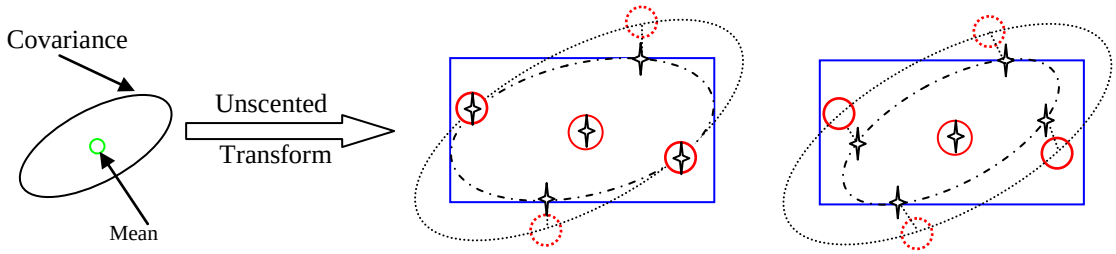
Estimated state covariance P_k: $P_k = P_k^- - K_k P_k^{y-} K_k^T$

3. Constraints Handling in UKF

The generic UKF algorithm considers nothing about constraints, but constraints handling can still be done within different steps of it, such as the generated sigma points, the propagated sate, the predicted state, or the estimated state, *etc*.

The sigma points violating the constraints will fall out of the feasible region. An intuitive method is to pull the sigma points back into the feasible region by some proper transforms, e.g., by sigma points projection or scaling, as illustrated in Figure 2. In Figure 2, the blue square represents the feasible region of the constrained state. The red circles are the original generated sigma points, and the black stars are the sigma points after constraints handling. The dashed ellipse forms the profile of the original sigma points, while the dash dotted ellipse is the profile of rearranged sigma points. In the "projection" method, only those sigma points that are out of the feasible region are projected onto the constraint surface, while in the "scaling" method all the sigma points are scaled together to fit the constraint surface.The original sigma points distribute symmetrically on an elliptical sphere, with the previously estimated state being the center. We'd better preserve the symmetry after the transform.

Another method is not to change the positions of the sigma points, but to modify the weights of them to make the predicated state satisfying the constraints.



Original state mean and covariance scaling (a) sigma points projection (b) sigma points

Figure 2. Illustration of the Two Techniques for Sigma Points Rearrangement in the UKF

3.1. Sigma Points Projection

In the sigma points projection technique, the sigma points violating the constraints are projected onto the constraint surface, while the positions of the other sigma points are unchanged, as shown in Figure 2(a). The projection is done by replacing the sigma point with a point on the surface that has a smallest distance from the sigma point. The advantage of projection is that both the mean and covariance are computed based on the constraint information, making the state estimate more accurate. If it is an equality constraint, *i.e.*, $g(x_k) = d_{k-1}$, or it is an inequality constraint but the constraint feasible region is a close set, say $g(x_k) \le d_{k-1}$, the sigma points violating the constraints can be simply projected onto the constraint surface $g(x_k) = d_{k-1}$; if the feasible region is an open set, e.g., $g(x_k) < d_{k-1}$, the sigma points violating the constraints can not be projected onto the surface directly, and they should be projected onto an interior surface, $g(x_k) = d_{k-1} - \varepsilon$ with $\varepsilon > \mathbf{0}$, in the feasible region.

The projection approach is computationally efficient, but it may have poor performance for nonlinear constrained system, as shown in [10].

3.2. Sigma Points Scaling

In the sigma points scaling technique, all the sigma points are scaled together to fit the constraint surface if there exist some sigma points being out of the constraint surface, as shown in Figure 2(b). The scaling can be done in an anisotropic way or in an isotropic way.

In the anisotropic scaling manner, only the sigma points that go beyond the feasible region are scaled onto the constraint surface by multiplying different sigma points with different positive coefficients, and the coefficients may also vary at different iterations. In the isotropic scaling manner, all the sigma points are scaled together to make them fit the constraint surface if any of the generated sigma points go beyond the feasible region of the constraint.

The sigma points scaling can be done using the scaled unscented transform (SUT) [16] with the parameter α not to be 1. Constraints can be handled very well if a proper α is chosen. The problem of SUT is that the estimated state covariance P_k may not be positive semi-definite, since the weight w_c^0 for computing covariance matrices may be negative when $\alpha < 1$.

3.3. Sigma Points Re-Weighting

As aforementioned, constraints can also be handled with the predicated state, which relies on not only the sigma points, but also the weights w_m. As is shown in

(4), w_m is controlled by the three free parameters α, β and κ, so we can change the value of them to modify w_m. One thing worthy of mentioning is that when we change the value of α and κ, we modify not only the weights w_m of the sigma points, but also the locations of the sigma points, so it is actually a composition of sigma points re-weighting and scaling.

Many heuristics have been developed to help set these three parameters. One of the heuristics for setting $\boldsymbol{\theta} = (\alpha, \beta, \kappa)$ is that $\beta = 2$ is optimal if the state distribution is exactly Gaussian [16]. And it is often recommended to set the parameters to be $\alpha = 1, \beta = 0$ and $\kappa = 3 - n$ for general non-Gaussian systems [17]. On the other hand, arbitrary parameters may cause the UKF to have embarrassingly poor performance for nonlinear problems, because the predictive variances can be far too small if the sigma points are placed in unlucky locations [18].

4. Parameter Learning for Projection based UKF with GPO

As aforementioned, the sigma points projection based method may have poor state estimation performance for nonlinear constrained systems, while the sigma points scaling based method may fail due to the negative-definite estimated state covariance. On the other hand, a combination of these two techniques may have good performance since a positive semi-definite estimated state covariance can be achieved in the projection based UKF if a proper α is chosen. Extending this idea further, we propose in this paper a projection based UKF algorithm with optimized parameter vector $\boldsymbol{\theta} = (\alpha, \beta, \kappa)$ to solve the constrained state estimation problem. The parameters are optimally learnt with GPO.

4.1. GPO based Parameters Learning for the Generic UKF

GPO based parameters learning for the generic UKF was firstly proposed by R. Turner and C. E. Rasmussen in [18]. The key idea is to interpret the UKF as a model, not merely an approximation method. This interpretation can then allow us to learn the free parameters in a model based manner from training data. The proposed model based learning algorithm can solve the so called "sigma point collapse" problem for the generic UKF, which can result in a significant increase in predictive performance over default settings of the parameters in the UKF.

GPO utilizes a Gaussian process based optimizer to find the optimal model parameter $\boldsymbol{\theta}$, which amounts to finding a maximum of a structured function $l(\boldsymbol{\theta})$, usually the log marginal likelihood with T training observations:

$$l(\boldsymbol{\theta}) = \log p(y_{1:T} \mid \boldsymbol{\theta}) = \sum_{t=1}^{T} \log p(y_t \mid y_{1:t-1}, \boldsymbol{\theta}) \tag{5}$$

A maximization strategy that trades-off exploration with exploitation takes into account both the mean function $E[l(\boldsymbol{\theta})]$ and the posterior variance function $Var[l(\boldsymbol{\theta})]$, and the optimizer is programmed to evaluate the maxima of

$$J(\boldsymbol{\theta}) = E[l(\boldsymbol{\theta})] + C\sqrt{Var[l(\boldsymbol{\theta})]} \tag{6}$$

where C is a constant to control the exploration exploitation trade-off.

GPO treats optimization as a sequential decision problem and assumes we provide a feasible set of $\boldsymbol{\theta}$ to search within. At each step it uses its posterior over the objective function J to look for the best $\boldsymbol{\theta}$. The detailed Gaussian process optimizer algorithm can be found in [18].

4.2. Parameters Learning for Constrained UKF

The original GPO based parameters learning algorithm considers nothing about constraint handling. But constraints should be considered and handled during the

parameters learning process. The learnt parameters with and without constraints concerned may differ greatly, and the learnt parameters without constraints concerned will be improper for constrained state estimation, as illustrated in section V.

In order to handle constraints, the original GPO based parameters learning algorithm for the generic UKF is extended to the projected UKF. That is, the sigma-points-projection technology is utilized to handle constraints during the parameters learning process. We summarize the proposed method in Algorithm 1.

Algorithm 1. Parameters learning for Projection based UKF with GPO

1. Generate a series of observations according to the process function and the measurement function as the training data.
2. Choose a proper range of $Rg_\theta = [\alpha_l, \alpha_h] \times [\beta_l, \beta_h] \times [\kappa_l, \kappa_h]$.
3. Build a multi-grid $\boldsymbol{\theta} = \{\boldsymbol{\theta}_i, i = 1, 2, ..., M\}$ with M grid points in Rg_θ.
4. **for** $p = 1$ to M
5. Calculate $Y = \{l(\boldsymbol{\theta}_p)\}$ according to the projection based UKF with $\{\boldsymbol{\theta}_p\}$ as the parameters, using the training data.
6. **for** $q = 1$ to C (C is the number of evaluated candidates)
   Sample a random initial point $s_0 \sim N(E(\boldsymbol{\theta}_p), \mathrm{cov}(\boldsymbol{\theta}_p))$;
   Maximize the criterion J w.r.t $\boldsymbol{\theta}_*$, while $\boldsymbol{\theta}_*$ initialized at s_0: $(s_q, F_q) \leftarrow \max J$.
7. **end for**
8. Append $s(\arg\max F)$ to $\boldsymbol{\theta}$
9. Append $l(s(\arg\max F))$ to Y
10. **end for**
11. **return** $\boldsymbol{\theta}(\arg\max Y)$

In Algorithm 1, α_l and α_h are the permitted lowest value and the permitted highest value for the parameter α respectively. The notations $\beta_l, \beta_h, \kappa_l, \kappa_h$ have similar meanings. $N(E(\boldsymbol{\theta}_p), \mathrm{cov}(\boldsymbol{\theta}_p))$ is the normal distribution with a mean of $E(\boldsymbol{\theta}_p)$ and a covariance of $\mathrm{cov}(\boldsymbol{\theta}_p)$.

One thing worthy of mentioning is about the implementation of the algorithm. During the parameters learning process, α may vary from 0 to 1 and the estimated state covariance P_k may not be positive semi-definite, which will interrupt the normal iterations of the UKF algorithm and result in an unsuccessful learning of the parameter. The augmented UKF [16] can help in part to solve the problem, in which the augmented state vector x_k^a and the augmented state covariance matrix P_k^a are defined as $x_k^a = \begin{bmatrix} x_k & \varepsilon_k & v_k \end{bmatrix}^T$ and $P_k^a = diag(P_k, Q_k, R_k)$ respectively.

The proposed parameters learning process is then carried on to the new constrained augmented state estimation problem.

5. Constrained State Estimation Simulations

After the optimized parameters are obtained, constrained state estimation can be achieved with the same projection based UKF.

As an example, we consider the gas-phase, reversible reaction [12]. The discrete model of the system is

$$x_{1,k+1} = \frac{x_{1,k}}{1 + 2k_r \Delta t\, x_{1,k}}, \quad x_{2,k+1} = x_{2,k} + \frac{k_r \Delta t\, x_{1,k}}{1 + 2k_r \Delta t\, x_{1,k}}$$

The state vector is $\boldsymbol{x} = [P_A \quad P_B]^T = [x_1 \quad x_2]^T$, and the output is $y = [1 \quad 1]\boldsymbol{x}$. P_A and P_B are the partial pressures. $k_r = 0.16$ and Δt is the integration step length. Assume the system experiences Gaussian noise both in the states and the outputs, and the noise have zero means and covariance of Q_k and R_k.

The parameters used for this system are

$$\Delta t = t_{k+1} - t_k = 0.1,\ \boldsymbol{x}_0 = [3 \quad 1]^T,\ \hat{\boldsymbol{x}}_0 = [0.1 \quad 4.5]^T$$

$$P_0 = diag(6^2, 6^2),\ Q_k = diag(0.001^2, 0.001^2),\ R_k = 0.1^2.$$

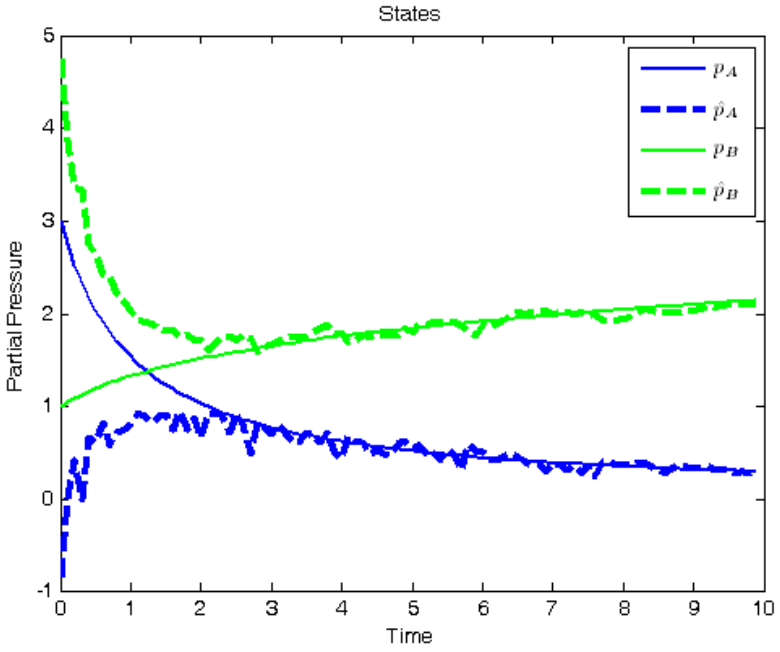
The constraints are such that $\hat{\boldsymbol{x}}_k \geq \boldsymbol{0}$.

As Figure 3(a) shows, the unconstrained UKF with default parameters fails to converge to the true state. State estimation under the UKF with default parameters and constraints handled by sigma points projection is shown in Figure 3(b). As Figure 3(b) shows, the state can converge to the true state, but it takes more than 10 samples for it to converge. If the pre-learnt optimal parameters $(\alpha, \beta, \kappa) = (0.0825, 0.4775, 0.3150)$ is used, the state can converge much faster (only about 2 samples), as shown in Figure 3(c). For a comparison, we also show the result from Kolås S. *et al* [12] in Figure 3(d). Our method has smaller estimation errors at the initial part. We also measured the average square root error of the true and estimated state by 100 Monte Carlo simulations. The average square root estimation errors for the four algorithms are $[0.659, 0.658]^T$, $[0.443, 0.453]^T$, $[0.055, 0.071]^T$, $[0.40, 0.49]^T$ respectively, which shows that our proposed method is superior to the others.

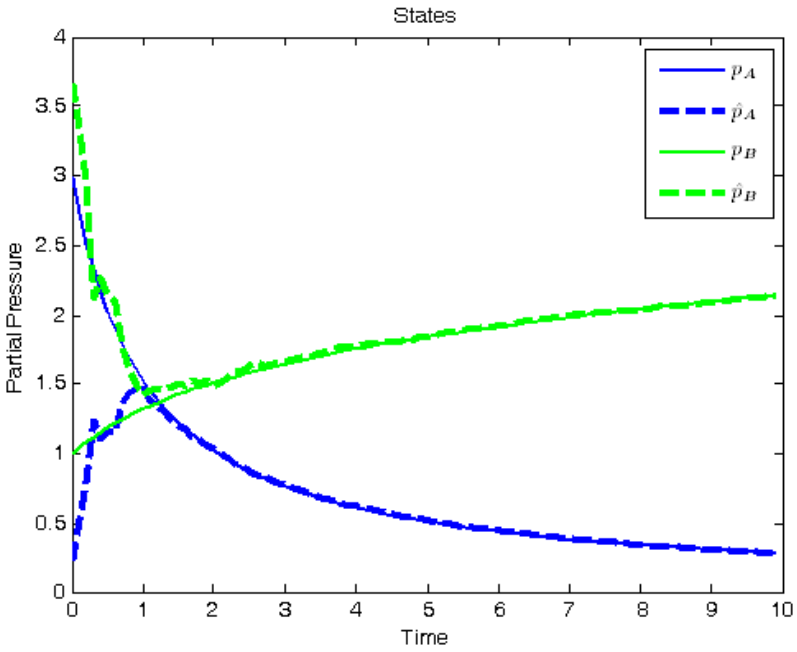
On the other hand, during the sate estimation process, our proposed method needs no optimization procedure and so takes less time to estimate the constrained states.

6. Conclusions

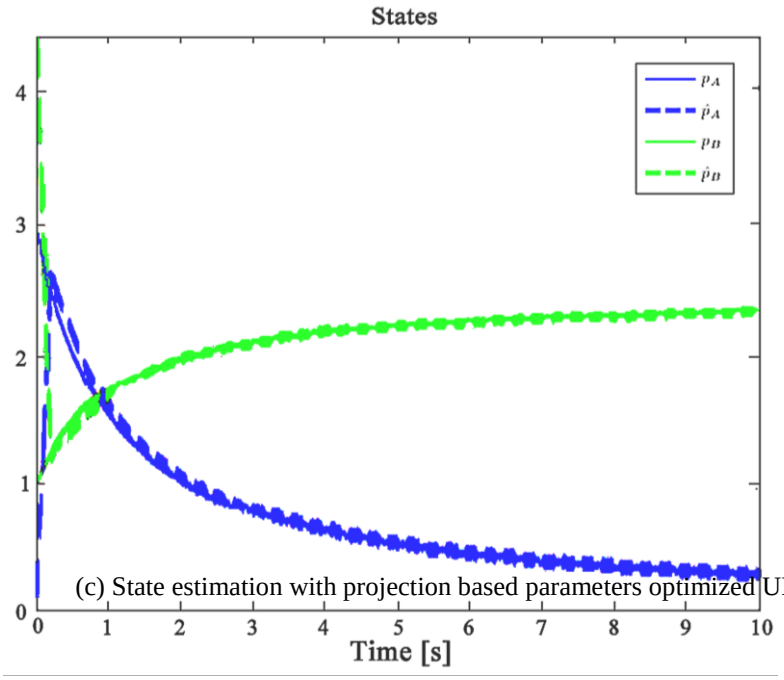
A UKF based constrained state estimation method is proposed in this paper. The three free parameters of the UKF are first optimally learnt based on a GPO method, with the constraints embedded and handled by sigma points projection. The state estimation is



(a) State estimation under unconstrained UKF with default parameters.



(b) State estimation under UKF with default parameters, with constraints handled by sigma points projection.



(c) State estimation with projection based parameters optimized UKF.

(d) State estimation with the method of Kolås S *et al.*

Figure 3. Simulations for the Two-State Batch Reaction Problem

then done with the projection based optimized parameters UKF. The handling of the constraints in the estimation process can be regarded as a combination of sigma points projection, scaling and re-weighting. Simulation results on different systems show that the proposed method is effective and efficient.

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