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Harmonic Extension on an LB-Fractal

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Abstract

The intent of the paper is to establish the energy relationship on adjacent graphs of LB-fractal constructed iteratively through an iterated function system containing five contraction maps. The harmonic matrices on the LB-fractals are obtained through harmonic extension. The normal derivatives are also found for the same.

Keywords: *Harmonic Extension; Graph Energy; Normal Derivative; Line Based Fractal*

1. Introduction

The analysis on fractals was initiated by Kigami [5] to study heat diffusion and vibration problems in materials having fractal structures. Strichartz [13] studied the dynamical aspects of fractals through differential equations. One way to study the fractals, particularly the class of nested fractals and their physical properties, is to consider their Laplacian. Thus one of the main aims of analysis in general is construction of a Laplacian which, in most of the cases, is an operator with specific properties related to an energy form. Due to non smooth structures of fractals, it was very difficult to define differential operator on it. The study of diffusion processes on fractals was initiated independently by Goldstein [3] and Kusuoka [14]. They use probabilistic approach to define a Brownian motion on fractals. Kigami [5] introduced Laplacian on Sierpinski Gasket through analytic approach by using Dirichlet forms. Zhang and Feng [17] found the graph energy on Sierpinski Gasket and interpolation curve. This study was further enriched by many authors (see, for instance [13], [16] and several references thereof). Begue et al. [2] studied the Laplacian on the Sierpinski carpet and defined normal derivatives on the boundary of it. For a detailed study on fractals and their applications, one may refer Barnsley [1] (see also [6-12], [15] and several references therein). In this paper, our aim is to construct a line based fractal (LB-fractal) in R^2 by the method given in [1] using deterministic algorithm on the IFS. We compute graph energy for the constructed fractal and find harmonic matrices on the LB-fractal with the help of harmonic extension on the curve.

2. Preliminaries

First, we start with the basic concepts required for our work.

Throughout the paper, V_m is used for the set of all vertices at *m*-level lines, $u \mid V_0$ means; *u* confines on V_0 and $l(V) = \{ f \text{ such that } f : V \to R \}$.

Definition 2.1 [13]. Let *G* be a finite connected graph and u be a real function on its vertices, then the graph energy $E_G(u)$ is defined as follows

$$E_G(u) = \sum_{x \sim y} (u(x) - u(y))^2. \quad (1)$$

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The associated bilinear form of it is given by $E_G(u,v) = \sum_{x \sim y} (u(x) - u(y))(v(x) - v(y))$.

Here $x \sim y$ implies that x and y are the adjacent vertices of the graph.

Definition 2.2 [4]. For any $\rho \in l(V_0)$, there exists a unique $u \in F$, such that $u \mid V_0 = \rho$ and $\varepsilon(u,u) = \min\{\ \varepsilon(v,v) : v \in F, v \mid V_0 = \rho\}$. Furthermore, u is the unique solution of

$$\begin{cases} (H_m v)|V_m \setminus V_0 = 0; \quad \text{for all} \quad m \geq 1 \\ \qquad v \mid V_0 = \rho, \end{cases} \qquad (2)$$

where H_m is the Laplacian on V_m. The function u given above is known as harmonic function.

Let u be a harmonic function on V_m.. Then the extension of u onto V_{m+1} which minimizes the energy is called harmonic extension of u and denoted by $\tilde{u}$.

We assume that the following condition holds:

$$E_{m+1}(u) = c_m E_m(u), \qquad (3)$$

where c_m is the renormalization factor.

Definition 2.3 [4]. Renormalized graph energy ε_m is defined as

$$\varepsilon_m(u) = \alpha^{-m} E_m(u) \quad (4)$$

Definition 2.4 [4]. Let $x \in V_0$ and u be a continuous function on some subset of R^2. We say that the normal derivative $\partial_n u(x)$ exists if limit of $r^{-m} \sum_{q \in V_{m,p}} (u(p) - u(q))$ exists and it is given by

$$\partial_n u(x) = \lim_{m \to \infty} r^{-m} \sum_{q \in V_{m,p}} (u(p) - u(q)) \quad (5)$$

3. Construction of Line Based Fractal

We take a line segment L_0 in R as an initiator and apply the following contraction transformations on it iteratively.

$$f_1\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$f_2\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1/3 \\ 1/3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1/3 \\ 0 \end{bmatrix}$$

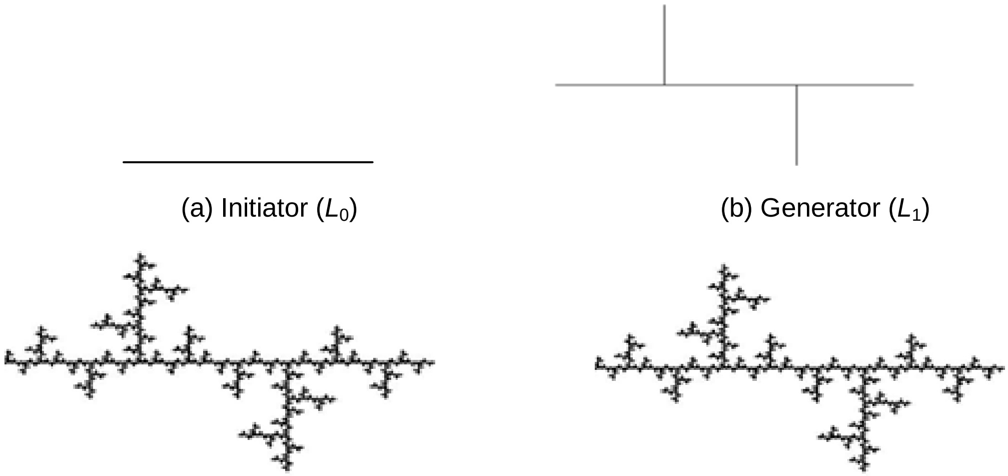
$$f_3\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1/3 \\ 0 \end{bmatrix},$$

$$f_4\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1/3 \\ -1/3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2/3 \\ 0 \end{bmatrix},$$

and

$$f_5\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2/3 \\ 0 \end{bmatrix},$$

where $x, y \in R$. Some steps for generation of the line based fractal (LB-Fractal) for twenty iterations are shown in Figure 1.



(a) Initiator (L_0) (b) Generator (L_1)

(c) After 10 iterations (d) After 20 iterations

Figure 1. Generation of LB-Fractal

4. Graph Energies and Harmonic Extension

Let $V_0 = \{X, Y\}$, $u(X) = u_X$ and $u(Y) = u_Y$ in Figure 1(a). To calculate c_m in equation (3), we extend the function u to V_1. Values of the extension $\tilde{u}$ at the nodes are such that the energy is minimized with respect to $\tilde{u}$. Here

$$E_0(u) = (u_X - u_Y)^2 \quad (6)$$

Let V_1={X, Y, A, B, C, D}, $\tilde{u}(A) = \tilde{u}_A$, $\tilde{u}(B) = \tilde{u}_B$ and $\tilde{u}(C) = \tilde{u}_C$ and $\tilde{u}(D) = \tilde{u}_D$ at first iteration. Then

$$E_1(\tilde{u}) = (\tilde{u}_A - u_X)^2 + (\tilde{u}_A - \tilde{u}_B)^2 + (\tilde{u}_B - \tilde{u}_C)^2 + (\tilde{u}_B - \tilde{u}_D)^2 + (\tilde{u}_B - \tilde{u}_Y)^2 \quad (7)$$

We minimize the energy $E_1(\tilde{u})$ with respect to $\tilde{u}_A$, $\tilde{u}_B$, $\tilde{u}_C$ and $\tilde{u}_D$. By calculus, to minimize, we equate the partial derivatives of $E_1(\tilde{u})$ with respect to $\tilde{u}_A$, $\tilde{u}_B$, $\tilde{u}_C$ and $\tilde{u}_D$ to zero. We have

$$2\tilde{u}_A - \tilde{u}_B - u_X = 0 \quad (8)$$

$$2\tilde{u}_B - \tilde{u}_A - u_Y = 0 \quad (9)$$

$$\tilde{u}_A = \tilde{u}_C \quad (10)$$

$$\tilde{u}_B = \tilde{u}_D \quad (11)$$

$$\tilde{u}_A = \tilde{u}_C = \frac{2u_X + u_Y}{3}; \ \tilde{u}_B = \tilde{u}_D = \frac{u_X + 2u_Y}{3} \quad (12)$$

Substituting these values in (7), we have

$$E_1(\tilde{u}) = \left(\frac{2u_X - u_Y}{3} - u_X\right)^2 + \left(\frac{2u_X + u_Y}{3} - \frac{u_X + 2u_Y}{3}\right)^2 + \left(\frac{u_X + 2u_Y}{3} - u_Y\right)^2$$
$$= \frac{1}{3}(u_X - u_Y)^2 = \frac{1}{3}E_0(u)$$

The same idea applies for the extension from V_1 to V_2 and the values of the points in $V_2 \backslash V_1$ also satisfy "$\frac{2}{3} - \frac{1}{3}$" rule.

$$E_2(u') = \left(\frac{1}{3}\right)^2 E_0(u) \quad (13)$$

In general, we have extension from V_m to V_{m+1}. So we have

$$E_{m+1}(u') = \left(\frac{1}{3}\right)^{m+1} E_0(u) \quad (14)$$

Here 1/4 is called renormalization factor. Then the renormalization energy is

$$\varepsilon_m(u) = \left(\frac{1}{3}\right)^{-m} E_m(u); \qquad \varepsilon_1(u) = \left(\frac{1}{3}\right)^{-1} E_1(u) = \varepsilon_0(u) \quad (15)$$

and in general,

$$\varepsilon_{m+1}(u) = \left(\frac{1}{3}\right)^{-(m+1)} E_{m+1}(u) = \varepsilon_m(u)$$

Finally, we calculate the harmonic matrices for $i = 1$ to 5, using the following equation

$$\begin{pmatrix} u(f_i(u_X)) \\ u(f_i(u_Y)) \end{pmatrix} = A_i \begin{pmatrix} u_X \\ u_Y \end{pmatrix}. \quad (16)$$

We know that

$$f_1(u_X) = u_X,\ f_2(u_X) = \tilde{u}_C,\ f_3(u_X) = \tilde{u}_A,\ f_4(u_X) = \tilde{u}_D,\ f_5(u_X) = \tilde{u}_B;$$

$$f_1(u_Y) = \tilde{u}_A,\ f_2(u_Y) = \tilde{u}_A,\ f_3(u_Y) = \tilde{u}_B,\ f_4(u_Y) = \tilde{u}_B,\ f_5(u_Y) = u_Y.$$

We calculate harmonic matrices using (16) as:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 2/3 & 1/3 \end{bmatrix}\begin{bmatrix} u_X \\ u_Y \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix}\begin{bmatrix} u_X \\ u_Y \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}\begin{bmatrix} u_X \\ u_Y \end{bmatrix}, \qquad A_4 = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{bmatrix}\begin{bmatrix} u_X \\ u_Y \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 1/3 & 2/3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_X \\ u_Y \end{bmatrix}.$$

5. Normal Derivative

Using definition 2.4, normal derivative of LB-fractal can be written as

$$\partial_n u(x) = \lim_{m \to \infty} \left(\frac{1}{3}\right)^{-m} \sum_{q \in V_{m,p}} (u(p) - u(q));$$

and normal derivative at boundary point $p_i\,(p_i \in V_0)$ is

$$\partial_n u(x) = \lim_{m \to \infty} \left(\frac{1}{3}\right)^{-m} \sum_{q \in V_{m,p}} (u(p_i) - u(p_{i+1})).$$

6. Conclusion

We establish the relationship of energy on adjacent graphs of LB- fractal iteratively and thus obtain a sequence of renormalized energy, which has a non decreasing pattern. So, we can conclude that LB-fractal has finite energy. We also obtain Normal derivative and harmonic matrices on LB-fractal using concept of harmonic extension.

References

[1] M. F. Barnsley, Fractals Everywhere, Academic Press, San Diego, **(1988)**.
[2] M. Begue, T. Kalloniatis, and R. S. Strichartz, Harmonic functions and the spectrum of the Laplacian on the Sierpinski carpet, Fractals 21:1 **(2013)**. ArXiv:1201.5136
[3] S. Goldstein, Random walks and diffusions on fractals, IMA Math. Appl, vol. 8, pp. 121–129 **(1987)**.
[4] J. Kigami, Analysis on Fractals, Machinery Industry Press, Beijing, **(2004)**.
[5] J. Kigami, A harmonic calculus on the Sierpinski spaces, Japan J. Appl. Math, vol. 6, pp. 259–290 **(1989)**.
[6] B. Prasad, Fractals for A-iterated function and multifunction, International Journal of Applied Engineering Research vol.7, no.11, 2032–2036 **(2012)**.
[7] B. Prasad and K. Katiyar, Fractals via Ishikawa iteration, Communications in Computer and Information Science, vol.140, pp 197-203 **(2011)**.
[8] B. Prasad, B. Singh and K. Katiyar, A method of curve fitting by recurrent fractal interpolation, Proceedings of International Conference in Computational Intelligence, **(2012)** December 18-20; Tamilnadu, India, pp. 1–4.
[9] B. Prasad, B. Singh and K. Katiyar, A hidden variable fractal interpolation surface method, Proceedings of international conference on Information and Mathematical Sciences, **(2013)** October 24-26; Punjab, India, pp. 156–158.

[10] B. Prasad, B. Singh and K. Katiyar, Modeling curves via fractal interpolation with VSFF, Proceedings of International Conference on Advances in Computer Engineering and Applications, **(2014)** March 19–20; Uttar Pradesh, India.
[11] B. Prasad and K. Katiyar, Stability and fractal patterns of complex logistic map Cybernetics and Information Technologies, vol. 14, no. 3, pp. 14–24 **(2014)**.
[12] B. Prasad and K. Mishra, Fractals in G-metric spaces, Applied Mathematical Sciences, vol. 7, no. 109, pp. 5409 - 5415 **(2013)**.
[13] R. Strichartz, Diffrential Equations on Fractals, Princeton University Press, **(2006)**.
[14] S. Kusuoka, A diffusion process on a fractal, Probabilistic Methods in Mathematical Physics, Academic Press, Boston, pp. 251–274 **(1987)**.
[15] S. L. Singh and B. Prasad, A. Kumar, Fractals via iterated functions and multifunctions, Chaos Solitons and Fractals, vol. 39, pp. 1224–1231 **(2009)**.
[16] R. Uthayakumar and A. Nalayini Devi, Harmonic extension on level-3 Sierpinski gasket via electrical network, Proceedings of the International Conference on Information Communication and Embedded Systems, **(2013)** February 21–22; Tamilnadu, India, pp. 998–1004.
[17] B. Zhang and Z. Feng, Graph energy on fractal interpolation curve, International Journal of Nonlinear Science, vol. 10, pp. 253–256 **(2010)**.

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