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A New Fundamental and Numerical Method for the Fractional Partial Differential Equations

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Abstract

Fractional order has the characteristics of memory and non-locality and it is different with integer order. Therefore, fractional differential equations can be used to describe some abnormal natural phenomena. At the same time, how to solve the fractional order partial differential equation and differential equations with fractional order has become a very important research field. Besides analytic solution, it is also important to investigate the numerical methods for fractional differential equations. In the paper, fundamental solution of the time fractional partial differential equation has been deduced, which is derived by Furrier transform and Laplace transform. According to the simulation, there is little difference between numerical solution and the exact solution when the solution is the time variable function. The results show the validity of the method.

Keywords: *fractional partial differential function, fundamental solution, numerical solution, comparison*

1. Introduction

With the development of research on natural science and economics, deficiencies of integer order differential equations [1] has been revealed in the simulation of concrete problems. Then, it is needed to explore and study the mathematical tool which has more general significance. Among them, fractional calculus [2-4] is one of the mathematical problems worthy of further study. Fractional derivative [5-6] with real number as the order is defined by the integral method, and it is the characteristics of memory and non-locality [7-8], which is different with integer order. Therefore, compared with the integer order differential equation, fractional differential equations [9-10] can meet the special needs in some conditions.

Fractional partial differential equation can be used to describe some abnormal natural phenomena [11-13]. Fractional diffusion equation can be used to describe the abnormal slow diffusion phenomenon [14-16] in porous media with fractal structures, and fractional advection diffusion equation can be used to describe the anomalous osmotic phenomenon of fluid in medium [17-18]. Therefore, research on fractional partial differential equation application attracts more and more extensive attention, especially on the study of fractional diffusion equation [19-20] and fractional advection diffusion equation [21-22].

With the wide application of fractional calculus system, how to solve the fractional order partial differential equation and differential equations with fractional order has become a very important research field. Laplace transform and Mellin transform can be used to find the fundamental solution of the time fractional advection-diffusion equation in the space of half plane [23]. By using Furrier transform and Laplace transform, fundamental solution of time fractional diffusion equation of the whole plane can be deduced. The fundamental solution is obtained through the Laplace inverse transform of Mittag-Leffler function.

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However, it is similar as integer order differential equation, and fractional solutions of only a few fractional order differential equations can be found by analytical means. So, it is also important to investigate the numerical methods for fractional differential equations. At the same time, due to the special characteristic of fractional differential equations, calculation of fractional differential equations will need great workload, and how to get the effective numerical methods in solving fractional differential equations is an important problem.

The main work of the paper is to deduce the fundamental solution of the time fractional partial differential equation [24-25]. It includes some conditions: the time fractional diffusion equation, time fractional reaction-diffusion equations, the time fractional advection-diffusion equation and their corresponding integer order partial differential equation. The fundamental solution is derived by Furrier transform and Laplace transform. The main contribution of the paper is to get a fundamental solution. In the paper, we also use a numerical method to calculate the solution of the function. The remainder of the paper is shown as the following: the function to be solved is shown in Section 2. Preliminary knowledge is described in Section 3. Fundamental solution is expressed in Section 4. The experiment is shown in Section 5 and the conclusion is described in Section 6.

2 The Function

The function we considered can be shown as the following:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = -\lambda^2 u(x,t) - v\frac{\partial u(x,t)}{\partial x} + D\frac{\partial^2 u(x,t)}{\partial x^2} \tag{1}$$

$$0 < \alpha \le 1$$

$$u(x,0) = f(x) \qquad -\infty < x \le \infty$$

$$u(\pm\infty, t) = 0 \qquad t > 0$$

Where, $\lambda \ge 0$, $v \ge 0$, $D > 0$, $\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}}$ is the Caputo Fractional Partial Differential Equations, and the definition can be describes as follows:

$$\frac{\partial^{\alpha} f(t)}{\partial t^{\alpha}} = \begin{cases} \dfrac{d^n f(t)}{dt^n} & \alpha = n \in Z \\ \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} \dfrac{d^n f(\tau)}{d\tau^n} d\tau, & n-1 < \alpha < n \end{cases} \tag{2}$$

When $v \ne 0$, the equation can be transformed as the following to get the results easily:

$$u(x,t) = C(\xi,t) e^{v\xi/(2\sqrt{D})}, \quad \xi = \frac{x}{\sqrt{D}} \tag{3}$$

Then the equation (1) can be described as follows:

$$\frac{\partial^{\alpha} C(\xi,t)}{\partial t^{\alpha}} = -m^2 C(\xi,t) + \frac{\partial^2 C(\xi,t)}{\partial \xi^2} \tag{4}$$

Where, $m^2 = \frac{v^2}{4D} + \lambda^2$.

Here, we just take condition of $v = 0$ into consideration, that is to say that the function is expressed as the following:

$$\frac{\partial^{\alpha} C(x,t)}{\partial t^{\alpha}} = -\lambda^{2} C(x,t) + D\frac{\partial^{2} C(x,t)}{\partial x^{2}}, \qquad x \in \Omega, t > 0 \tag{5}$$

3. Preliminary Knowledge

(1) Laplace transform definition

$$F(s) = L\{f(t); s\} = \int_{0}^{\infty} f(t) dt \tag{6}$$

And its inverse transformation is the following:

$$F(t) = L^{-1}\{f(s); t\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st} f(s) dt \qquad c = \mathrm{Re}(s) \tag{7}$$

(2) Fourier transform definition

$$H(w) = F\{h(t); w\} = \int_{-\infty}^{\infty} e^{iwt} h(t) dt \tag{8}$$

And its inverse transform is the following:

$$h(t) = F^{-1}\{H(w); t\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iwt} H(w) dw \tag{9}$$

(3) Definition of Caputo fractional derivative

$${}_{0}^{c}D_{t}^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_{0}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d(\tau) & n-1 < \alpha < n \\ \\ \dfrac{d^{(n)} f(t)}{dt^{n}} & \alpha = n \end{cases} \tag{10}$$

And the Laplace transform is as the following:

$$L\{{}_{0}^{c}D_{t}^{\alpha} f(t); s\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0) \qquad n-1 < \alpha < n \tag{11}$$

(4) Wright function

$$W(-z; \alpha, \beta) = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\alpha k + \beta) k!}. \tag{12}$$

There is also a definition as follows:

$$M_{\alpha}(Z) = \sum_{k=0}^{\infty} \frac{(-1)^{k} z^{k}}{\Gamma(-\alpha k + (1-\alpha)) k!} \tag{13}$$

It is defined as $M_{\alpha}(z)$, and then, if $\alpha = 1$, $M_{1}(z) = \delta(z-1)$. Where, $\delta(z)$ is Dirac δ function.

4 The basic Solution of the Time Fractional Partial Differential Equations

Some definition:

$$u(x,t) = C(\zeta, t)\exp\left(\frac{v\zeta}{2\sqrt{D}}\right) \tag{14}$$

$$\lambda^2 = (v^2 / 4D) + m^2 \tag{15}$$

If we give the formula as follows:

$$E_\alpha(ct^\alpha) \xleftrightarrow{L} \frac{s^{\alpha-1}}{s^\alpha - c}, \qquad \mathrm{Re}(s) > |c|^{\frac{1}{\alpha}} \tag{16}$$

Where, $c \in C$, and E_α is a function.

$$E_\alpha = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \qquad \alpha > 0, z \in C \tag{17}$$

Then, we can get:

$$\hat{u}(k,t) = E_\alpha\left[-(Dk^2 + \lambda^2)t^\alpha\right]\hat{f}(k) \tag{18}$$

With the inverse Fourier transform,

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ikx} E_\alpha\left[-(Dk^2 + \lambda^2)t^\alpha\right]\hat{f}(k)dk \\ &= \int_{-\infty}^{+\infty} G_C^{(\alpha)}(x-y,t) f(y)dy \end{aligned} \tag{19}$$

Where,

$$G_C^{(\alpha)}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ikx} E_\alpha\left[-(Dk^2 + \lambda^2)t^\alpha\right]dk \tag{20}$$

This is called the Green function or the basic solution. It should meet the demand of the following condition:

$$G_C^{(\alpha)}(k,t) = E_\alpha[-(Dk^2 + \lambda^2)t^\alpha] \tag{21}$$

And

$$\overline{G}(k,s) = \frac{s^{\alpha-1}}{s^\alpha + (Dk^2 + \lambda^2)} \tag{22}$$

This equation can be transformed to the following form:

$$\overline{G}(k,s) = E\left[-(Dk^2 + \lambda^2)t\right] \tag{23}$$

Consider the Fourier and Laplace relationships:

$$\frac{1}{2\sqrt{\pi}} u^{-\frac{1}{2}} e^{-\frac{x^2}{4u}} \xleftrightarrow{F} e^{-uk^2} \tag{24}$$

And

$$t^{-\alpha} M_\alpha\left(\frac{u}{t^\alpha}\right) \xleftrightarrow{L} s^{\alpha-1} e^{-us^\alpha} \tag{25}$$

Then the Green Function can be described as:

$$G_C(x,t) = \frac{1}{2\sqrt{\pi}}(Dt)^{-\frac{1}{2}} e^{\frac{-x^2}{4Dt}} e^{-\lambda^2 t} \tag{26}$$

$$G(x,t)=\frac{t^{-\alpha}}{2\sqrt{\pi}}\int_0^\infty e^{\frac{-x^2}{4u}}e^{-\lambda^2 u}u^{-\frac{1}{2}}M_\alpha(\frac{u}{t^\alpha})du \tag{27}$$

If set $u=\kappa t^\alpha$, then

$$\begin{aligned}G(x,t)&=\frac{1}{2\sqrt{\pi t^\alpha}}\int_0^\infty e^{\frac{-x^2}{4t^\alpha\kappa}-\lambda^2 t^\alpha\kappa}\kappa^{-\frac{1}{2}}M_\alpha(\kappa)d\kappa\\&=\frac{1}{2\sqrt{\pi t^\alpha}}\int_0^\infty e^{\frac{-x^2}{4t^\alpha\kappa}-(\frac{v^2}{4D}+\lambda^2)t^\alpha\kappa}\kappa^{-\frac{1}{2}}M_\alpha(\kappa)d\kappa\end{aligned} \tag{28}$$

If $\lambda=0$, it will be the time fractional advection-dispersion function. Then $G(x,t)$ can be described as:

$$G(x,t)=\frac{1}{2\sqrt{\pi t^\alpha}}\int_0^\infty e^{\frac{-x^2}{4t^\alpha\kappa}-\frac{v^2}{4D}t^\alpha\kappa}\kappa^{-\frac{1}{2}}M_\alpha(\kappa)d\kappa \tag{29}$$

If $\alpha=1$, the function will be the integer partial differential equation, and the $G(x,t)$ can be described as:

$$\begin{aligned}G(x,t)&=\frac{1}{2\sqrt{\pi t}}\int_0^\infty e^{\frac{-x^2}{4t\kappa}-(\frac{v^2}{4D}+\lambda^2)t\kappa}\kappa^{-\frac{1}{2}}M_1(\kappa)d\kappa\\&=\frac{1}{2\sqrt{\pi t}}\int_0^\infty e^{\frac{-x^2}{4t\kappa}-(\frac{v^2}{4D}+\lambda^2)t\kappa}\kappa^{-\frac{1}{2}}\delta(\kappa-1)d\kappa\\&=\frac{1}{2\sqrt{\pi t}}e^{\frac{-x^2}{4t}-(\frac{v^2}{4D}+\lambda^2)t}\end{aligned} \tag{30}$$

When $\lambda=0$, the $G(x,t)$ can be described as:

$$\begin{aligned}G(x,t)&=\frac{1}{2\sqrt{\pi t}}\int_0^\infty e^{\frac{-x^2}{4t\kappa}-(\frac{v^2}{4D}+\lambda^2)t\kappa}\kappa^{-\frac{1}{2}}M_1(\kappa)d\kappa\\&=\frac{1}{2\sqrt{\pi t}}\int_0^\infty e^{\frac{-x^2}{4t\kappa}-(\frac{v^2}{4D}+\lambda^2)t\kappa}\kappa^{-\frac{1}{2}}\delta(\kappa-1)d\kappa\\&=\frac{1}{2\sqrt{\pi t}}e^{\frac{-x^2}{4t}-\frac{v^2 t}{4D}}\end{aligned} \tag{31}$$

When $v=0$, $D>0$, the $G(x,t)$ can be described as:

$$G(x,t)=\frac{1}{2\sqrt{\pi t^\alpha}}\int_0^\infty e^{\frac{-x^2}{4t^\alpha\kappa}-\lambda^2 t^\alpha\kappa}\kappa^{-\frac{1}{2}}M_\alpha(\kappa)d\kappa \tag{32}$$

If $\alpha = 1$, the $G(x,t)$ can be described as:

$$G(x,t) = \frac{1}{2\sqrt{\pi t}} e^{\frac{-x^2}{4t} - \lambda^2 t} \tag{33}$$

When $\lambda = 0$, $v = 0$, $D > 0$, the $G(x,t)$ can be described as:

$$G(x,t) = \frac{1}{2\sqrt{\pi t^{\alpha}}} M_{\frac{\alpha}{2}}(\kappa)\left(\frac{|x|}{\sqrt{D t^{\alpha}}}\right) \tag{34}$$

Then if $\alpha = 1$, the $G(x,t)$ can be described as:

$$G(x,t) = \frac{1}{2\sqrt{\pi}} e^{\frac{-x^2}{4Dt}} \tag{35}$$

5. Comparison of the Results

Equation (36) and equation (37) are solved by the method proposed in the paper. Besides, the numerical method [26-27] is also used to solve the both of the two equations. In addition, the boundary value condition is transferred into homogeneous condition, and the first 15 orders are adopted. The solution of the two methods is compared in the figure 1 to figure 10. In all the Figures, the line is the exact solution and the "*" represents the numerical solution.

Equation 1:

$$\begin{cases} \dfrac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} + \lambda^2 u(x,t) - D\dfrac{\partial^2 u(x,t)}{\partial x^2} = 0 & x \in (0,L), t > 0 \\ u(x,0) = f(x), & x \in (0,L), \\ u(0,t) = u(L,t) = 0, & t > 0 \end{cases} \tag{36}$$

Equation 2:

$$\begin{cases} \dfrac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} + \lambda^2 u(x,t) - D\dfrac{\partial^2 u(x,t)}{\partial x^2} = 0 & x \in (0,L), t > 0 \\ u(x,0) = f(x), & x \in (0,L), \\ u_x(0,t) = u_x(L,t) = 0, & t > 0 \end{cases} \tag{37}$$

Figure 1 to Figure 4 show the comparison of the numerical solution and the exact solution of equation 1 when the solution is the time variable function. It can be seen from the figures that there is little difference between the numerical solution and the exact solution. Figure 5 to Figure 8 show the comparison of the numerical solution and the exact solution of equation 2. The numerical results also have little difference to the exact solution.

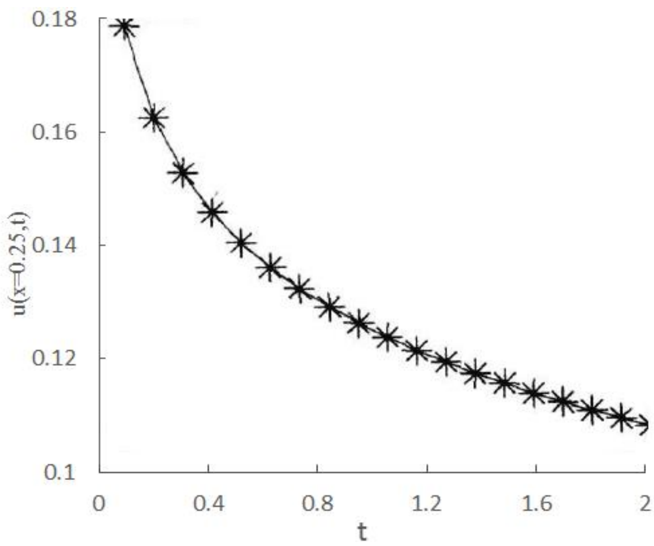

Figure 1. *λ*=1/8, *x*=0.25, *α*=0.25

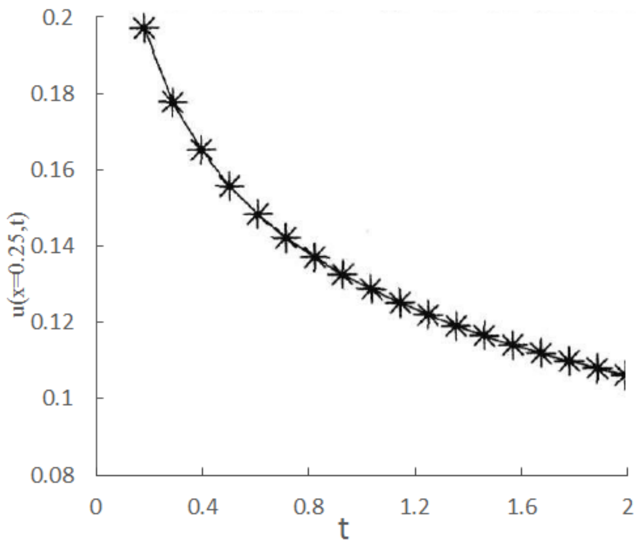

Figure 2. *λ*=1/8, *x*=0.25, *α*=0.5

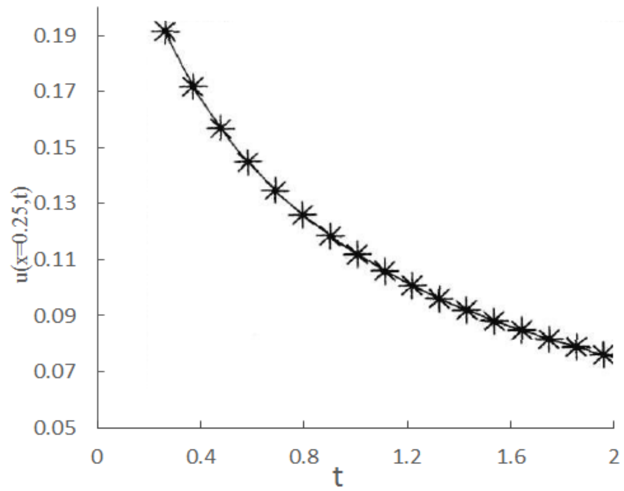

Figure 3. *λ*=1/8, *x*=0.25, *α*=0.75

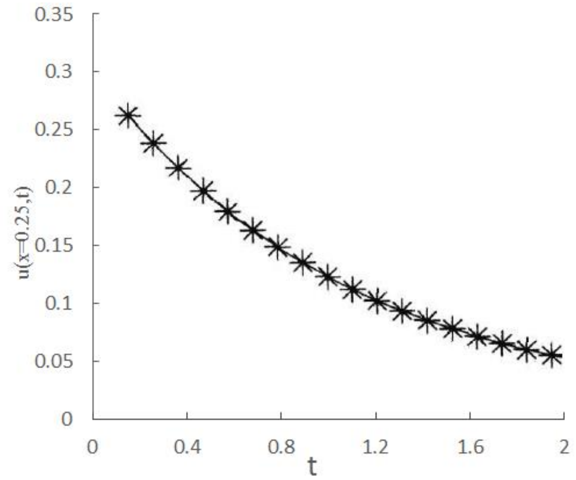

Figure 4. ***λ*****=1/8,** ***x*****=0.25,** ***α*****=1**

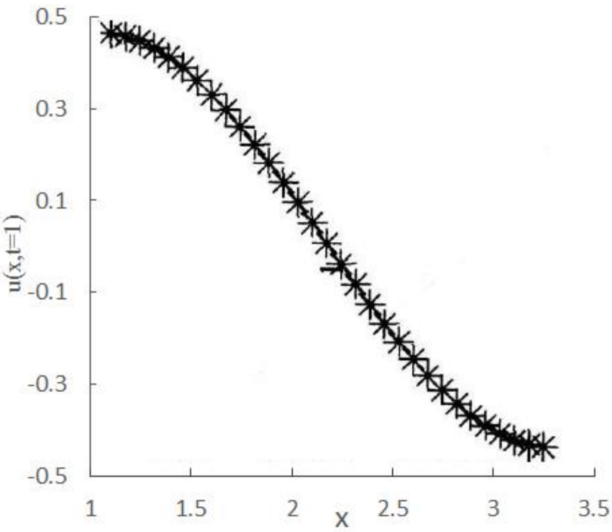

Figure 5. ***λ*****=1/8,** ***α*****=0.25,** ***t*****=1**

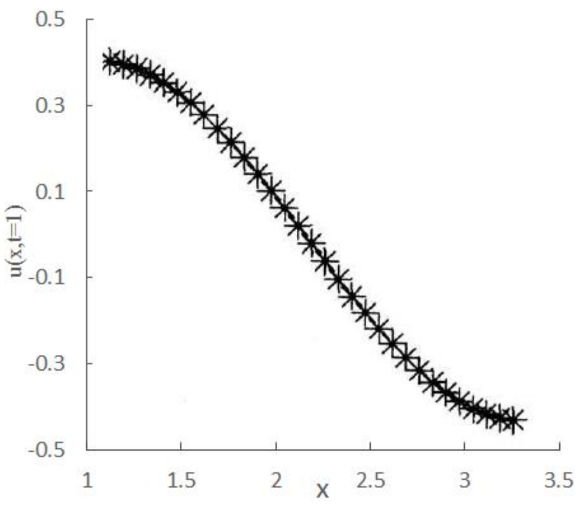

Figure.6 ***λ*****=1/8,** ***α*****=0.5,** ***t*****=1**

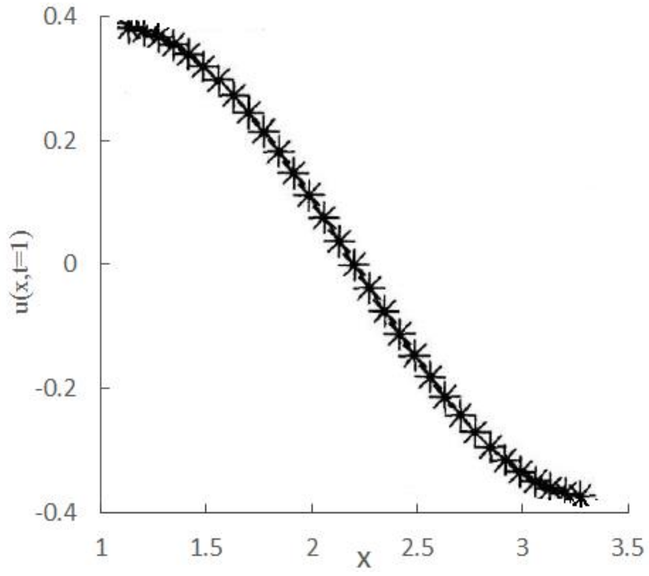

Figure 7. *λ*=1/8, *α*=0.75, *t*=1

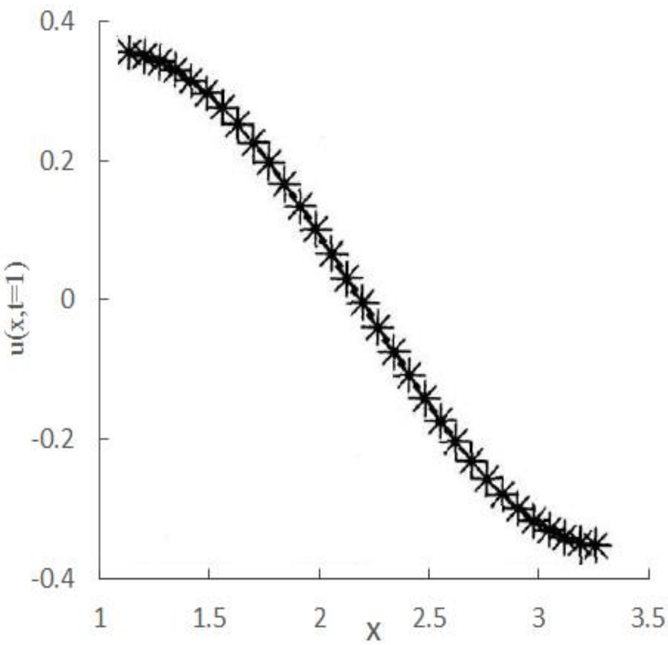

Figure 8. *λ*=1/8, *α*=1, *t*=1

Figure 9 and 10 shows the comparison of numerical solution and the exact solution of equation 1 when the solution is the time variable function. It can be seen from the figures that there is little difference between numerical solution and the exact solution. When t is small, value of solution increases with the increasing of α , and when t is big enough, value of solution decreases with the increasing of α .

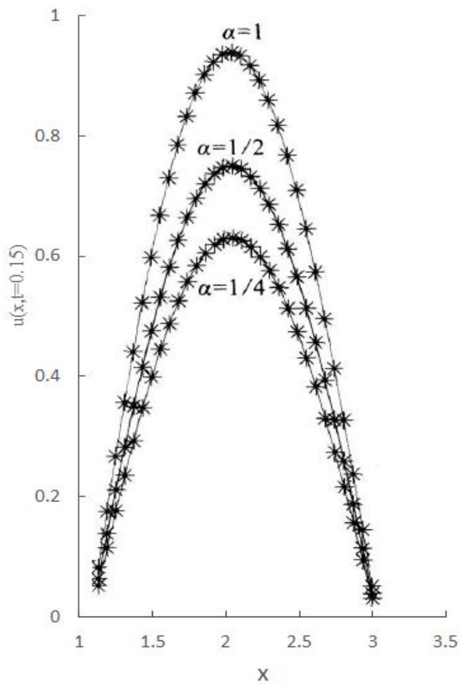

Figure 9. λ=1/8, *t*=0.15

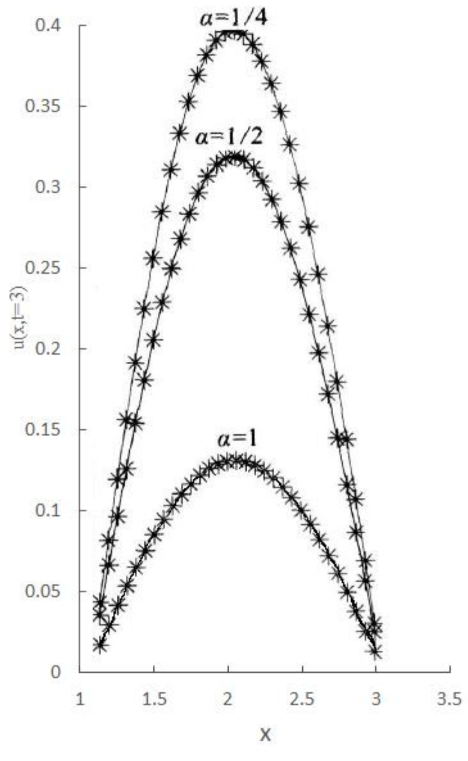

Figure 10. λ=1/8, *t*=3

6. Conclusion

Development of natural science and economics, deficiencies of integer order differential equations has been found. Fractional calculus has more general significance in describing some special problems, and it is one of the mathematical problems worthy of further study. Fractional order has the characteristics of memory and non-locality and it is different with integer order. Therefore, fractional differential equations can meet the special needs in some conditions, and it can be used to describe some abnormal natural phenomena. At the same time, how to solve the fractional order partial differential equation and differential equations with fractional order have become a very important research field. It is also important to investigate the numerical methods for fractional differential equations. How to get effective numerical methods in solving fractional differential equations is an important problem.

In the paper, fundamental solution of the time fractional partial differential equation has been deduced, which is derived by Furrier transform and Laplace transform. According to the simulation verification, validity of the method has been proved. There is little difference between numerical solution and the exact solution when the solution is the time variable function. Numerical solution and the exact solution decrease with the attenuation of time t . When t is small, value of solution is increases with the increasing of α , and when t is big enough, value of solution decreases with the increasing of α .

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