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Analytical Exact Solution of Advection Problems via Coupling of Sumudu Decomposition Method and Taylor's Series

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Abstract

In this paper, we applied Sumudu decomposition method coupled with the Taylors series to solve linear and non-linear Advection problems. It is observed that the proposed method is highly suitable for such problems and overcomes some of the basic deficiencies of traditional decomposition method. Several examples are given to re-confirm the efficiency of the suggested algorithm.

Keywords: *Advection problems; Nonlinear problems; He's polynomials; Taylors series*

1. Introduction

The rapid development of nonlinear sciences witnesses a wide range of analytical and numerical techniques by various scientists. Most of the developed schemes have their limitations like limited convergence, divergent results, linearization, discretization, unrealistic assumptions and non-compatibility with the versatility of physical problems [1-21]. The basic motivation of present study is the extension of traditional Modified Decomposition Method coupled with the Taylor's series [10, 11] to tackle linear and nonlinear advection problems which arise very frequently in applied and engineering sciences. It has been observed that the coupling of decomposition method with Taylor's series enhances its efficiency and reduces the computational work to a tangible level. Moreover, this version is more user-friendly and it overcomes the complexities of selection of initial value. Several examples are given which reveal the efficiency and reliability of the proposed algorithm.

2. Definitions and Properties of the S–Transform

In early 90's, Watugala [21] introduced a new integral transform, named the Sumudu transform and applied it to the solution of ordinary differential equation in control engineering problems. Sumudu transform is defined over the set of the following functions

$$A=\{ f(t):\tau_1,\tau_2>0,\left|f(t)\right|<Me^{t/\tau_j}, if\ t\in(-1)^j\times[0,\infty)\} \tag{1}$$

By the following formula

$$G(u)=S[f(t);u]:=\int_0^\infty f(ut)e^{-t}dt,\ u\in(-\tau_1,\tau_2). \tag{2}$$

Sumudu transforms of the derivatives of (x) is

$$S[\frac{d^nU}{dx^n}]=\frac{1}{u^n}S[U(x)]-\frac{1}{u^n}U(0)-\frac{1}{u^{n-1}}U'(0)-...-\frac{U^{n-1}(0)}{u}. \tag{3}$$

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Some special properties of the Sumudu transform are as follows:

1. $S[1]=1;$
2. $S[\frac{t^n}{\Gamma(n+1)}]=u^n,\ n>0;$
3. $S[e^{at}]=\frac{1}{1-au};$
4. $s[\alpha f(x)+\beta g(x)]=\alpha S[f(x)]+\beta S[g(x)].$
5. Other properties of the Sumudu transform can be found in [22].

3. Method of Analysis

We consider the general inhomogeneous nonlinear equation with initial conditions given below:

$$LU+RU+NU=h(x,t), \tag{4}$$

Where L is the highest order derivative which is assumed to be easily invertible, R is a linear differential operator of order less than L, NU represents the nonlinear terms and $h(x,t)$ is the source term. First we explain the main idea of SDM: the method consists of applying Sumudu transform

$$S[LU]+S[RU]+S[NU]=S[h(x,t)]. \tag{5}$$

Using the differential property of Sumudu transform and initial conditions we get

$$\frac{1}{u^n}S[u(x,t)]-\frac{1}{u^n}u(x,0)-\frac{1}{u^{n-1}}u'(x,0)-...-\frac{u^{n-1}(x,0)}{u}+S[RU]+S[NU]=S[h(x,t)] \tag{6}$$

By arrangement we have

$$S[u(x,t)]=u(x,0)+uu'(x,0)+...+u^{n-1}u^{n-1}(x,0)-u^nS[RU]-u^nS[NU]+u^nS[h(x,t)] \tag{7}$$

The second step in Sumudu decomposition method is that we represent solution as an infinite series:

$$u(x,t)=\sum_{i=0}^{\infty}p^iu_i(x,t). \tag{8}$$

and the nonlinear term can be decomposed as

$$NU(x,t)=\sum_{i=0}^{\infty}H_i, \tag{9}$$

Where H_i are He's Polynomials of U_0, U_1,... U_n and it can be calculated by formula

$$H_i=\frac{1}{i!}\frac{d^i}{dp^i}[N\sum_{i=0}^{\infty}p^iu_i]_{p=0},\ i=0,1,2,... \tag{10}$$

Thus, we have

$$S\left[\sum_{i=0}^{\infty}p^iU_i(x,t)\right]=U(x,0)+uU'(x,0)+...+u^{n-1}U^{n-1}(x,0)$$
$$-u^2S[RU]-u^nS[RU(x,t)]-u^nS[\sum_{i=0}^{\infty}H_i]+u^nS[h(x,t)]. \tag{11}$$

On comparing both sides and by using standard SDM we have:

$$S[U_0(x,t)]=U(x,0)+uU'(x,0)+...+u^{n-1}U^{n-1}(x,0)+u^nS[h(x,t)]=K(x,u) \tag{12}$$

Then it follows that

$$S[U_1(x,t)] = -u^n S[RU_0(x,t)] - u^n S[H_0],$$
$$S[U_2(x,t)] = -u^n S[RU_1(x,t)] - u^n S[H_1]. \tag{13}$$

In more general, we have

$$S[U_{i+1}(x,t)] = -u^n S[RU_i(x,t)] - u^n S[H_i], i \geq 0. \tag{14}$$

On applying the inverse Sumudu transform

$$p^0; U_0(x,t) = K(x,t),$$
$$p^{i+1}; U_{i+1}(x,t) = -S^{-1}\left[u^n S\left[RU_i(x,t)\right] + u^n S\left[H_i\right]\right], \quad i \geq 0 \tag{15}$$

Where $K(x,t)$ represents the term that is arising from source term and prescribed initial conditions in the form of Taylor series.

4. Numerical Applications

Example 4.1 Consider the following partial differential equation

$$u_t + uu_x - u = e^t, \ t > 0, \tag{16}$$

with initial conditions,

$$u(x,0) = x+1. \tag{17}$$

Taking the Sumudu transform of Equation (16) with respect to t, we get

$$\frac{1}{u} S[U(x,t)] - \frac{1}{u} U(x,0) = S[e^t] + S[u - uu_x],$$
$$S\left[u(x,t)\right] = x + 1 + \frac{u}{1-u} + uS[u - u_x], \tag{18}$$

Taking the inverse Sumudu transform of Eq. (18) on both sides

$$u(x,t) = x + e^t + S^{-1}\left[uS[u - u_x]\right], \tag{19}$$

Eq. (19) we can write in the following form

$$\sum_{n=0}^{\infty} u_n(x,t) = x + e^t + S^{-1}\left[uS\left[\sum_{n=0}^{\infty} u_n - \sum_{n=0}^{\infty} u_n u_{nx}\right]\right], \tag{20}$$

Applying the Taylor series, we get

$$\sum_{n=0}^{\infty} u_n(x,t) = x + 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots + S^{-1}\left[uS\left[\sum_{n=0}^{\infty} u_n - \sum_{n=0}^{\infty} u_n u_{nx}\right]\right], \tag{21}$$

According to the proposed technique, we have the following recurrence relation

$$u_0(x,t) = x+1,$$
$$u_{n+1}(x,t) = f_{n+1}(x) + S^{-1}\left[uS[\sum_{n=0}^{\infty} u_n - \sum_{n=0}^{\infty} u_n u_{nx}]\right], n = 0,1,\ldots \tag{22}$$

Consequently, following approximation obtained,

$$u_0(x,t) = x+1,$$
$$u_1(x,t) = t + S^{-1}\left[uS[u_0 - u_0 u_{0x}]\right] = t,$$
$$u_2(x,t) = t + S^{-1}\left[uS[u_1 - u_1 u_{1x}]\right] = \frac{t^2}{2!},$$
$$\vdots \tag{23}$$

The series solution is

$$u(x,t) = x + 1 + t + \frac{t^2}{2!} + ..., \tag{24}$$

and the closed form is given by

$$u(x,t) = x + e^t. \tag{25}$$

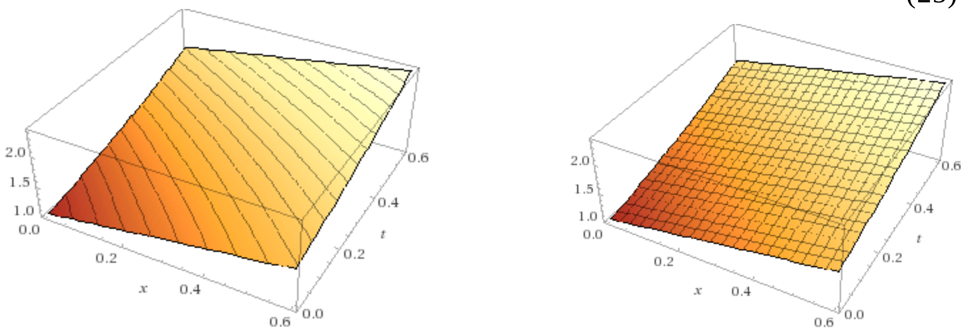


Figure 1. Graphical Representation of *u(x,t)* at Different Values of *x* and *t*

Example 4.2: Consider a nonlinear partial differential equation

$$u_{tt} + u^2 - u_x^2 = 0,\ t > 0, \tag{26}$$

with initial conditions

$$u(x,0) = 0,\ u_t(x,0) = e^x. \tag{27}$$

Taking Sumudu transform to Eq. (26) and (27), we obtain

$$\frac{1}{u^2} S[u(x,t)] - \frac{u(x,0)}{u^2} - \frac{u_t(x,0)}{u} = S[u_x^2 - u^2]. \tag{28}$$

$$S[u(x,t)] = ue^x + u^2 S[u_x^2 - u^2]. \tag{29}$$

By applying the inverse Sumudu transform, we get

$$u(x,t) = te^x + S^{-1}\left[u^2 S[u_x^2 - u^2]\right] \tag{30}$$

Which assumes a series solution of the function $u(x,t)$ and is given by

$$U(x,t) = \sum_{i=0}^{\infty} p^i u_i(x,t), \tag{31}$$

Thus we have

$$\sum_{i=0}^{\infty} p^i u_i(x,t) = te^x + S^{-1}\left[u^2 S[\sum_{i=0}^{\infty} H_i(u) - \sum_{i=0}^{\infty} B_i(u)]\right], \tag{32}$$

H_i And B_i are He's polynomials that represent nonlinear terms. Apply the Taylor series, we get

$$\sum_{i=0}^{\infty} p^i u_i(x,t) = t(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...) + S^{-1}\left[u^2 S[\sum_{i=0}^{\infty} H_i(u) - \sum_{i=0}^{\infty} B_i(u)]\right], \tag{33}$$

Thus by comparing the coefficients of p , we get

$$p^0; u_0(x,t) = t,$$

$$p^1; u_1(x,t) = tx + S^{-1}\left[u^2 S[u_{0x}^2 - u_0^2]\right] = tx - \frac{t^4}{12},$$

$$p^2; u_2(x,t) = \frac{tx^2}{2!} + S^{-1}\left[u^2 S[2u_{0x}u_{1x} - 2u_0 u_1]\right] = \frac{tx^2}{2!} - \frac{xt^4}{6} + \frac{t^7}{252},$$

$$\vdots \tag{34}$$

Therefore we have series solution,

$$u(x,t) = t + tx - \frac{t^4}{12} + \frac{tx^2}{2!} - \frac{xt^4}{6} + \frac{t^7}{252} + \ldots, \tag{35}$$

Therefore the exact solution when $p \to 1$ is

$$u(x,t) = te^x. \tag{36}$$

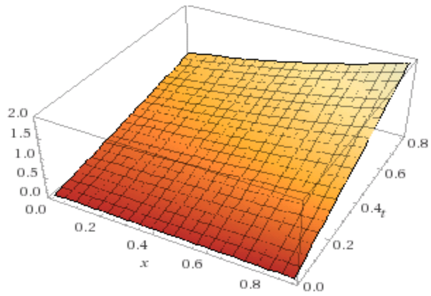


Figure 2. Graphical Representation of the *u(x,t)*

Example 4.3: Consider partial differential equation

$$u_t + \frac{1}{2}u_x^2 = e^x + t^2 e^{2x}, \tag{37}$$

with initial conditions,

$$u(x,0) = 0,$$

(38)

Taking the Sumudu transform of Equation (37) with respect to "*t*", we get

$$\frac{1}{u}S[U(x,t)] - \frac{U(x,0)}{u} = S[e^x] + S[t^2 e^{2x}] - S[\frac{1}{2}u_x^2], \tag{39}$$

$$S[U(x,t)] = ue^x + 2u^3 e^{2x} - uS[\frac{1}{2}u_x^2], \tag{40}$$

By applying the inverse Sumudu transform, we get

$$u(x,t) = te^x + \frac{t^3 e^{2x}}{3} - S^{-1}\left[uS[\frac{1}{2}u_x^2]\right], \tag{41}$$

Which assumes a series solution of the function $u(x,t)$ and is given by

$$U(x,t) = \sum_{i=0}^{\infty} p^i u_i(x,t), \tag{42}$$

Thus we have

$$\sum_{i=0}^{\infty} p^i u_i(x,t) = te^x + \frac{t^3 e^{2x}}{3} - S^{-1}\left[uS[\sum_{i=0}^{\infty} H_i(u)]\right], \tag{43}$$

where H_i are He's polynomials that represent nonlinear terms

Applying the Taylor series, we get

$$\sum_{i=0}^{\infty} p^i u_i(x,t) = t(1+x+\frac{x^2}{2!}+...)+\frac{t^3}{3}(1+2x+2x^2+...)-S^{-1}\left[uS[\sum_{i=0}^{\infty} H_i(u)]\right], \quad (44)$$

Thus by comparing the coefficients of p , we get

$$p^0; u_0(x,t) = t+\frac{t^3}{3},$$

$$p^1; u_1(x,t) = \left(t+\frac{2t^3}{3}\right)x,$$

$$p^2; u_2(x,t) = \left(t+\frac{4t^3}{3}\right)\frac{x^2}{2!},$$

$$\vdots \quad (45)$$

Therefore we have series solution,

$$u(x,t) = t+\frac{t^3}{3}+\left(t+\frac{2t^3}{3}\right)x+\left(t+\frac{4t^3}{3}\right)\frac{x^2}{2!}+..., \quad (46)$$

Therefore the exact solution when $p \to 1$ is

$$u(x,t) = te^x. \quad (47)$$

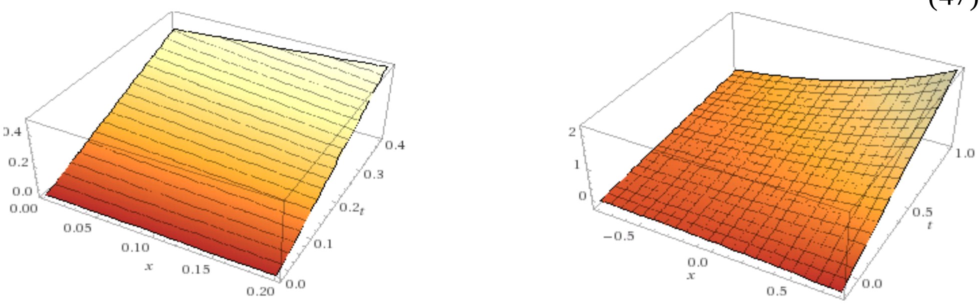


Figure 3. Graphical Representation of Solution *u(x,t)* for Different Values of *x* and *t*

Example 4.4 Consider the following partial differential equation with initial condition

$$u_t - \frac{1}{2}u_x^2 = -\sin(x+t) - \frac{1}{2}\sin 2(x+t), \quad (48)$$

$$u(x,0) = \cos x. \quad (49)$$

Taking the Sumudu transform of Equation (48), we get

$$\frac{1}{u}S[U(x,t)] - \frac{1}{u}U(x,0) = -S\left[\sin(x+t)\right] - \frac{1}{2}S\left[\sin 2(x+t)\right] + S\left[\frac{1}{2}u_x^2\right],$$

$$S[U(x,t)] = \cos x - \frac{u\sin x}{1+u^2} - \frac{u^2\cos x}{1+u^2} - \frac{u\sin 2x}{2(1+4u^2)} - \frac{u^2\cos 2x}{(1+4u^2)} + uS\left[\frac{1}{2}u_x^2\right], \quad (50)$$

By applying the inverse Sumudu transform we get

$$U(x,t) = \cos(x+t) + \frac{1}{4}\cos 2(x+t) - \frac{1}{4}\cos 2x + S^{-1}\left[uS\left[\frac{1}{2}u_x^2\right]\right], \quad (51)$$

Which assumes a series solution of the function $u(x,t)$ and is given by

$$U(x,t)=\sum_{i=0}^{\infty}p^{i}u_{i}(x,t), \tag{52}$$

Thus we have

$$\sum_{i=0}^{\infty}p^{i}u_{i}(x,t),=\cos(x+t)+\frac{1}{4}\cos 2(x+t)-\frac{1}{4}\cos 2x+S^{-1}\left[uS\left[\sum_{i=0}^{\infty}H_{i}(u)\right]\right], \tag{53}$$

where H_i are He's polynomials that represent nonlinear terms. Apply the Taylor series, we have

$$\sum_{i=0}^{\infty}p^{i}u_{i}(x,t),=\cos x+(-\sin x-\frac{1}{2}\sin 2x)t+(\frac{-1}{2}\cos x-\frac{1}{2}\cos 2x)t^{2}+S^{-1}\left[uS\left[\sum_{i=0}^{\infty}H_{i}(u)\right]\right], \tag{54}$$

Thus by comparing the coefficients of p , we get

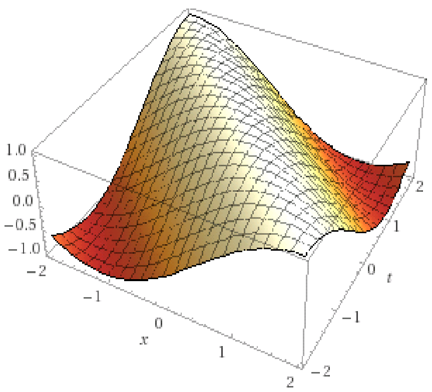
$$\begin{aligned} &p^{0};u_{0}(x,t)=\cos x,\\ &p^{1};u_{1}(x,t)=(-\sin x-\frac{1}{2}\sin 2x-\frac{1}{2}\sin^{2}x)t,\\ &p^{2};u_{2}(x,t)=\left(-\cos x-\cos 2x+\frac{1}{2}\sin x\cos x+\sin x\cos 2x-\sin^{2}x\cos x\right)\frac{t^{2}}{2!},\\ &\vdots \end{aligned} \tag{55}$$

Therefore we have series solution,

$$u(x,t)=\cos x(1-\frac{t^{2}}{2!}+\frac{t^{4}}{4!}-...)-\sin x(t-\frac{t^{3}}{3!}+\frac{t^{5}}{5!}-...), \tag{56}$$

Therefore the exact solution when $p\rightarrow 1$ is

$$u(x,t)=\cos(x+t). \tag{57}$$



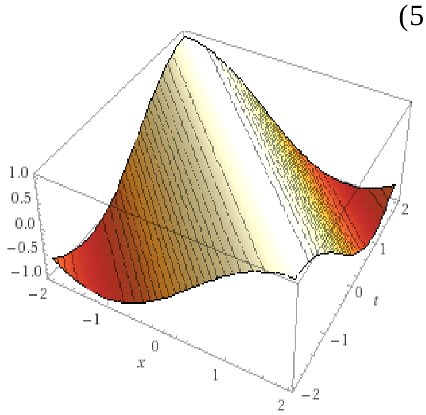


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Figure 4. Graphical Representation of Solution u(x,t) for Different Values of *x* and *t*

5. Conclusion

Sumudu Decomposition method coupled with Taylor's series is applied successfully to solve linear and nonlinear Advection problems. Moreover, suggested method makes the selection of initial values extremely simple and hence enhances its efficiency.

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