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Mond-Weir Duality for Multiobjective Programming with Invexity

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Abstract

This paper deals with the duality for a class of multiobjective programming problems including inequality constraints. To establish and prove the dual results for the multiobjective programming problems, the dual models and the classes of generalized invexity functions so-called $FJ-d-\alpha_\beta-\rho_\tau-\theta-$ *pseudoinvex-I are introduced. Using the new concepts, the weak dual, strong dual and converse dual theorems are obtained for the multiobjective programming problems.*

Keywords: *Multiobjective programming,* $FJ-d-\alpha_\beta-\rho_\tau-\theta-$ *pseudoinvex-I, weak duality, strong duality, converse duality*

1. Introduction

The term multiobjective programming is an extension of mathematical programming where a scalar valued objective function is replaced by a vector function. Many approaches for multiobjective programming problems have been explored in considerable details, see for example [1-4]. Furthermore, duality plays a fundamental role in mathematics, especially in optimization. It has not only used in many theoretical and computational developments in mathematical programming itself but also used in economics, control theory, business problems and other diverse fields. It is not surprising that duality is one of the important topics in multiobjective optimization. A large literature was developed around the duality in multiobjective fractional optimization under the generalized convexity assumption. Duality received more attention and many researchers have contributed to the development of duality in optimization. More specifically, Gao [5, 6] obtained several dual results for the multiobjective programming.

During the past decades, the common dualities were extended under the assumptions of generalized convexities. For example, we can see in [7-9]. In particular, the concept of second order generalized α - type I univex function were introduced and several duality theorems were obtained by Sharma and Gulaati [10] .We also can find another classes of generalized convex functions named as generalized second order $(F,\alpha,\rho,d)-$ convex functions introduced by Ahmad and Husain [11] and the duality results for Mond-Weir type vector dual were discussed. Also, we can see the references in Ref. [12 -14].

In this paper, motivated by the above work, we first introduce the new class of generalized invexity functions namely $FJ-d-\alpha_\beta-\rho_\tau-\theta-$ pseudoinvex-I ($FJ-d-\alpha_\beta-\rho_\tau-\theta-$ pseudoinvex-II et al.) by using the directional derivatives in the direction $\eta(x,\bar{x})$. Then the weak dual, strong dual and converse dual results are established and proved for the nondifferentiable nultiobjective programming problems under the assumptions of the new generalized convexities.

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2. Notations and Preliminaries

In this paper, we consider the following nondifferentiable multiobjective programming problem with constraints:

$$\text{(MP)}\quad \begin{array}{l} \text{Mininmize } f(x) = (f_1(x), f_2(x), \cdots, f_k(x)) \\ \text{subject to } g(x) = (g_1(x), g_2(x), \cdots, g_m(x)) \leqq 0 \\ \qquad x \in X \end{array}$$

Where $X \subseteq R^n$ is a nonempty open set, $f_i : X \to R (i = 1, 2, \cdots, k)$ and $g_j : X \to R(j = 1, 2, \cdots, m)$. Following, let us denote $I = \{1, 2, \cdots, k\}$ and $J = \{1, 2, \cdots, m\}$. Let $D = \{x \in X \mid g_j(x) \leqq 0, j \in J\}$ denote the set of all feasible solutions in the multiobjective programming problem (MP). Further, we denote by $J(x) = \{j \in J \mid g_j(x) = 0\}$ the index set of all active constraints of (MP) at an arbitrary feasible solution x, and $\overline{J}(x) = \{j \in J \mid g_j(x) < 0\}$.

The following convention for equalities and inequalities will be used the paper.

For any $x = (x_1, x_2, \cdots, x_n)^T, y = (y_1, y_2, \cdots, y_n)^T \in R^n$, we define:

$$\begin{array}{l} x = y \Leftrightarrow x_i = y_i, \ \forall \ i = 1, 2, \cdots, n, \\ x < y \Leftrightarrow x_i < y_i, \ \forall \ i = 1, 2, \cdots, n, \\ x \leqq y \Leftrightarrow x_i \leq y_i, \ \forall \ i = 1, 2, \cdots, n, \\ x \leq y \Leftrightarrow x \leqq y, \text{ there exists } i \text{ such that } x_i < y_i. \end{array}$$

Hereafter, we introduce some notions and definitions.

In the following definitions, $\eta(x, \overline{x}) : X \times X \to R^n$ is a vector valued function, with $\eta(x, \overline{x})$ nonzero.

Definition2.1. The directional derivative of f_i at $\overline{x} \in X$ in the direction $\eta(x, \overline{x})$, denoted $f_i'(\overline{x}; \eta(x, \overline{x}))$ is given by

$$f_i'(\overline{x}; \eta(x, \overline{x})) = \lim_{\lambda \to 0^+} \frac{f_i(\overline{x} + \lambda \eta(x, \overline{x})) - f_i(\overline{x})}{\lambda}, \ i \in \mathrm{I}\,.$$

Similarly, $g_j'(\overline{x}; \eta(x, \overline{x}))$ is denoted for $j \in J$.

Definition2.2[15] $w : X \to R$ is said to be semidirectionally differentiable at $\overline{x} \in X$, if there exists a nonempty subset $S \subset R^n$, such that $w'(\overline{x}; d)$ exists finite foe all $d \in S$. And w is said to be semidirectionally differentiable at $\overline{x} \in X$ in the direction $\eta(x, \overline{x})$, if its directional derivative $w'(\overline{x}; \eta(x, \overline{x}))$ exists finite for all $x \in X$.

By an extension of the previous definition, we say that a vector function $f = (f_1, f_2, \cdots, f_k) : X \to R^n$ is semidirectionally differentiable at $\overline{x} \in X$ in the direction $\eta(x, \overline{x})$, if each $f_i, i = 1, 2, \cdots, k$ is semidirectionally differentiable at $\overline{x} \in X$ in this direction. And simply, f is semidirectionally differentiable at $\overline{x} \in X$, if there exist a direction verifying the previous assertion.

Definition2.3. A feasible point $\overline{x}$ is said to be an efficient solution for (MP), if and only if there exists no another $x \in D$, such that

$$f(x) \leq f(\overline{x}).$$

Definition2.4. A feasible point $\overline{x}$ is said to be a weakly efficient solution for (MP), if and only if there exists no another $x \in D$, such that

$$f(x) < f(\bar{x}).$$

Let $f_i (i \in I)$ and $g_j (j \in J)$ be semidirectionally differentiable at $\bar{x} \in X$ in the direction $\eta(x, \bar{x})$, where, denote $\alpha, \beta : X \times X \to R_+ \setminus \{0\}$ $\rho = (\rho_1, \rho_2, \cdots, \rho_k) \in R^k$, $\tau_{J(\bar{x})} = \{\tau_j \in R,\ j \in J(\bar{x})\}$, θ:X × X $\to R^n$. Following we introduce new definitions for the pair of involved vector functions in (MP).

Definition2.5. (f, g) is said to be $FJ - d - \alpha_\beta - \rho_\tau - \theta -$ pseudoinvex-I (with respect to η) at $\bar{x} \in X$, if there exist $\alpha, \beta, \rho, \tau_{J(\bar{x})}$ and θ ,such that, for all $x \in X$, the following inequalities hold:

$$f(x) - f(\bar{x}) < 0 \Rightarrow \begin{cases} \alpha(x, \bar{x}) f'(\bar{x}; \eta(x, \bar{x})) + \rho \left\|\theta(x, \bar{x})\right\|^2 < 0, \\ \beta(x, \bar{x}) g'_{J(\bar{x})}(\bar{x}; \eta(x, \bar{x})) + \tau_{J(\bar{x})} \left\|\theta(x, \bar{x})\right\|^2 < 0. \end{cases}$$

Definition2.6. (f, g) is said to be $FJ - d - \alpha_\beta - \rho_\tau - \theta -$ pseudoinvex-II (with respect to η) at $\bar{x} \in X$, if there exist $\alpha, \beta, \rho, \tau_{J(\bar{x})}$ and θ ,such that, for all $x \in X$, the following inequalities hold:

$$f(x) - f(\bar{x}) \leq 0 \Rightarrow \begin{cases} \alpha(x, \bar{x}) f'(\bar{x}; \eta(x, \bar{x})) + \rho \left\|\theta(x, \bar{x})\right\|^2 < 0, \\ \beta(x, \bar{x}) g'_{J(\bar{x})}(\bar{x}; \eta(x, \bar{x})) + \tau_{J(\bar{x})} \left\|\theta(x, \bar{x})\right\|^2 < 0. \end{cases}$$

Definition2.7. (f, g) is said to be $FJ - d - \alpha_\beta - \rho_\tau - \theta -$ pseudoquasi-invex-I (with respect to η) at $\bar{x} \in X$, if there exist $\alpha, \beta, \rho, \tau_{J(\bar{x})}$ and θ ,such that, for all $x \in X$, the following inequalities hold:

$$f(x) - f(\bar{x}) \leq 0 \Rightarrow \begin{cases} \alpha(x, \bar{x}) f'(\bar{x}; \eta(x, \bar{x})) + \rho \left\|\theta(x, \bar{x})\right\|^2 < 0, \\ \beta(x, \bar{x}) g'_{J(\bar{x})}(\bar{x}; \eta(x, \bar{x})) + \tau_{J(\bar{x})} \left\|\theta(x, \bar{x})\right\|^2 \leqq 0. \end{cases}$$

Definition2.8. (f, g) is said to be $FJ - d - \alpha_\beta - \rho_\tau - \theta -$ pseudoquasi-invex-II (with respect to η) at $\bar{x} \in X$, if there exist $\alpha, \beta, \rho, \tau_{J(\bar{x})}$ and θ ,such that, for all $x \in X$, the following inequalities hold:

$$f(x) - f(\bar{x}) \leq 0 \Rightarrow \begin{cases} \alpha(x, \bar{x}) f'(\bar{x}; \eta(x, \bar{x})) + \rho \left\|\theta(x, \bar{x})\right\|^2 \leq 0, \\ \beta(x, \bar{x}) g'_{J(\bar{x})}(\bar{x}; \eta(x, \bar{x})) + \tau_{J(\bar{x})} \left\|\theta(x, \bar{x})\right\|^2 \leqq 0. \end{cases}$$

Definition2.9. (f, g) is said to be $FJ - d - \alpha_\beta - \rho_\tau - \theta -$ quasipseudo-invex-I (with respect to η) at $\bar{x} \in X$, if there exist $\alpha, \beta, \rho, \tau_{J(\bar{x})}$ and θ ,such that, for all $x \in X$, the following inequalities hold:

$$f(x) - f(\bar{x}) \leq 0 \Rightarrow \begin{cases} \alpha(x, \bar{x}) f'(\bar{x}; \eta(x, \bar{x})) + \rho \left\|\theta(x, \bar{x})\right\|^2 \leqq 0, \\ \beta(x, \bar{x}) g'_{J(\bar{x})}(\bar{x}; \eta(x, \bar{x})) + \tau_{J(\bar{x})} \left\|\theta(x, \bar{x})\right\|^2 < 0. \end{cases}$$

Definition2.10. (f, g) is said to be $FJ - d - \alpha_\beta - \rho_\tau - \theta -$ quasipseudo-invex-II (with respect to η) at $\bar{x} \in X$, if there exist $\alpha, \beta, \rho, \tau_{J(\bar{x})}$ and θ ,such that, for all $x \in X$, the following inequalities hold:

$$f(x) - f(\bar{x}) \leq 0 \Rightarrow \begin{cases} \alpha(x, \bar{x}) f'(\bar{x}; \eta(x, \bar{x})) + \rho \left\|\theta(x, \bar{x})\right\|^2 \leqq 0, \\ \beta(x, \bar{x}) g'_{J(\bar{x})}(\bar{x}; \eta(x, \bar{x})) + \tau_{J(\bar{x})} \left\|\theta(x, \bar{x})\right\|^2 \leq 0. \end{cases}$$

3. Mond-Weir Duality

In order to study the duality, we are going to tackle duality between the primal multiobjective problem (MP), and an associated problem of the Mond-Weir type. Now, let us formulate the dual problem of (MP) as follows.

$$\text{(MD)}\quad \begin{aligned} &\text{Max } f(u) \\ &\text{s.t. } \lambda^T f'(u;\eta(x,u)) + \mu^T g'(u;\eta(x,u)) \geqq 0, \quad \forall x \in X && (1) \\ &\mu_j g_j(u) = 0, \quad j \in J && (2) \\ &(\lambda,\mu) \geq 0, \quad u \in X && (3) \end{aligned}$$

Denote the set of all the feasible solutions of (MD) with

$$W = \left\{ \begin{array}{l} (u,\lambda,\mu) \in X \times R^k \times R^m : \lambda^T f'(u;\eta(x,u)) + \mu^T g'(u;\eta(x,u)) \geqq 0, \quad \forall x \in X \\ \mu_j g_j(u) = 0, j \in J, \ (\lambda,\mu) \geq 0 \end{array} \right\}$$

In the following, we shall establish the weak duality, strong duality and coverse duality result.

Theorem 3.1. (Weak Duality) Let x and (u,λ,μ) be feasible solutions for (MP) and (MD), respectively. If (f,g) is $FJ-d-\alpha_\beta-\rho_\tau-\theta-$ pseudoinvex-I at u with respect to η, with

$$\frac{\lambda^T \rho}{\alpha(x,u)} + \frac{\mu^T_{J(u)}\tau_{J(u)}}{\beta(x,u)} \geqq 0 \tag{4}$$

Then the following can not hold:

$$f(x) < f(u)$$

Proof: We proceed by contradiction. Suppose that

$$f(x) < f(u) \tag{5}$$

Using the feasible of (u,λ,μ) for (MD), we have

$$\lambda^T f'(u;\eta(x,u)) + \mu^T g'(u;\eta(x,u)) \geqq 0, \quad \forall x \in X$$

$$\mu_j g_j(u) = 0, j \in J$$

$$(\lambda,\mu) \geq 0$$

Which imply $(\lambda,\mu_{J(u)}) \geq 0, \mu_{\bar{J}(u)} = 0$, therefore

$$\sum_{i\in I} \lambda_i f_i'(u;\eta(x,u)) + \sum_{j\in J(u)} \mu_j g_j'(u;\eta(x,u)) \geqq 0 \tag{6}$$

On the other hand, from $FJ-d-\alpha_\beta-\rho_\tau-\theta-$ pseudoinvex-I at u of (f,g), the inequality (5) yields

$$\alpha(x,u) f'(u;\eta(x,u)) + \rho \left\|\theta(x,u)\right\|^2 < 0$$

$$\beta(x,u) g'_{J(u)}(u;\eta(x,u)) + \tau_{J(u)} \left\|\theta(x,u)\right\|^2 < 0$$

That is

$$\alpha(x,u) f_i'(u;\eta(x,u)) + \rho_i \left\|\theta(x,u)\right\|^2 < 0, \forall i \in I$$

$$\beta(x,u) g_j'(u;\eta(x,u)) + \tau \left\|\theta(x,u)\right\|^2 < 0, j \in J(u)$$

By the same $\lambda, \mu_{J(u)}$, with $\alpha(x,u) > 0, \beta(x,u) > 0$, the above two inequalities imply

$$\sum_{i\in I}\lambda_i f_i'(u;\eta(x,u)) + \sum_{j\in J(u)}\mu_j g_j'(u;\eta(x,u))$$
$$< -\left(\frac{\sum_{i\in I}\lambda_i\rho_i}{\alpha(x,u)} + \frac{\sum_{j\in J(u)}\mu_j\tau_j}{\beta(x,u)}\right)\|\theta(x,u)\|^2$$
$$= -\left(\frac{\lambda^T\rho}{\alpha(x,u)} + \frac{\mu_{J(u)}^T\tau_{J(u)}}{\beta(x,u)}\right)\|\theta(x,u)\|^2 \geqq 0$$

Which contradicts (6). This completes the proof.

Theorem 3.2 (weak duality) Let x and (u,λ,μ) be feasible solutions for (MP) and (MD), respectively. If (f,g) is $FJ-d-\alpha_\beta-\rho_\tau-\theta-$ pseudoinvex-II at u with respect to η , with

$$\frac{\lambda^T\rho}{\alpha(x,u)} + \frac{\mu_{J(u)}^T\tau_{J(u)}}{\beta(x,u)} \geqq 0 \tag{7}$$

Then the following can not hold:

$$f(x) \le f(u)$$

Proof: Using by contradiction. Suppose that

$$f(x) \le f(u)$$

From $FJ-d-\alpha_\beta-\rho_\tau-\theta-$ pseudoinvex-II at u of (f,g) , the above inequality yields

$$\alpha(x,u)f'(u;\eta(x,u)) + \rho\|\theta(x,u)\|^2 < 0$$
$$\beta(x,u)g_{J(u)}'(u;\eta(x,u)) + \tau_{J(u)}\|\theta(x,u)\|^2 < 0$$

That is

$$\alpha(x,u)f_i'(u;\eta(x,u)) + \rho_i\|\theta(x,u)\|^2 < 0\ ,\forall i\in I$$
$$\beta(x,u)g_j'(u;\eta(x,u)) + \tau_j\|\theta(x,u)\|^2 < 0,\ j\in J(u)$$

Since $\alpha(x,u)>0, \beta(x,u)>0$, the above two inequalities give

$$f_i'(u;\eta(x,u)) < -\frac{\rho_i}{\alpha(x,u)}\|\theta(x,u)\|^2\ ,\forall i\in I \tag{8}$$

$$g_j'(u;\eta(x,u)) < -\frac{\tau_j}{\beta(x,u)}\|\theta(x,u)\|^2,\ j\in J(u) \tag{9}$$

By the dual constraints (2)-(3), we know $(\lambda,\mu_{J(u)})\ge 0, \mu_{\bar{J}(u)}=0$. So, with the hypothesis (7), the inequalities (8)-(9) follow

$$\sum_{i\in I}\lambda_i f_i'(u;\eta(x,u)) + \sum_{j\in J(u)}\mu_j g_j'(u;\eta(x,u)) + \sum_{j\in \bar{J}(u)}\mu_j g_j'(u;\eta(x,u))$$
$$< -\left(\frac{\sum_{i\in I}\lambda_i\rho_i}{\alpha(x,u)} + \frac{\sum_{j\in J(u)}\mu_j\tau_j}{\beta(x,u)}\right)\|\theta(x,\bar{x})\|^2$$
$$= -\left(\frac{\lambda^T\rho}{\alpha(x,u)} + \frac{\mu_{J(u)}^T\tau_{J(u)}}{\beta(x,u)}\right)\|\theta(x,\bar{x})\|^2 < 0$$

That is

$$\lambda^T f'(u;\eta(x,u)) + \mu^T g'(u;\eta(x,u)) < 0$$

Which gives a contradiction to the constraints (1). This completes the proof.

Theorem 3.3.(weak duality) Let x and (u,λ,μ) be feasible solutions for (MP) and (MD), respectively. If (f,g) is $FJ-d-\alpha_{\beta}-\rho_{\tau}-\theta-$ pseudoquasi-invex-I at u with respect to η, and

$$\frac{\lambda^{T}\rho}{\alpha(x,u)}+\frac{\mu_{J(u)}^{T}\tau_{J(u)}}{\beta(x,u)}\geqq 0 \tag{10}$$

Then the following cannot hold:

$$f(x)\leq f(u)$$

Proof: we proceed by contradiction. Suppose that

$$f(x)\leq f(u)$$

Using $FJ-d-\alpha_{\beta}-\rho_{\tau}-\theta-$ pseudoquasi-invex-I at u of (f,g), the above inequality yields

$$\alpha(x,u)f'(u;\eta(x,u))+\rho\left\|\theta(x,u)\right\|^{2}<0$$

$$\beta(x,u)g'_{J(u)}(u;\eta(x,u))+\tau_{J(u)}\left\|\theta(x,u)\right\|^{2}\leqq 0$$

That is

$$\alpha(x,u)f_i'(u;\eta(x,u))+\rho_i\left\|\theta(x,u)\right\|^{2}<0\ ,\forall i\in I$$

$$\beta(x,u)g_j'(u;\eta(x,u))+\tau_j\left\|\theta(x,u)\right\|^{2}\leqq 0,\ j\in J(u)$$

By the dual constraints (2)-(3), we get $(\lambda,\mu_{J(u)})\geq 0,\mu_{\bar{J}(u)}=0$. So, with the hypothesis (10), $\lambda>0$ and $\alpha(x,u)>0,\beta(x,u)>0$, the two inequalities imply

$$\sum_{i\in I}\lambda_i f_i'(u;\eta(x,u))+\sum_{j\in J(u)}\mu_j g_j'(u;\eta(x,u))+\sum_{j\in\bar{J}(u)}\mu_j g_j'(u;\eta(x,u))$$

$$<-\left(\frac{\sum_{i\in I}\lambda_i\rho_i}{\alpha(x,u)}+\frac{\sum_{j\in J(u)}\mu_j\tau_j}{\beta(x,u)}\right)\left\|\theta(x,\bar{x})\right\|^{2}$$

$$=-\left(\frac{\lambda^{T}\rho}{\alpha(x,u)}+\frac{\mu_{J(u)}^{T}\tau_{J(u)}}{\beta(x,u)}\right)\left\|\theta(x,\bar{x})\right\|^{2}\leqq 0$$

That is

$$\lambda^{T}f'(u;\eta(x,u))+\mu^{T}g'(u;\eta(x,u))<0$$

Which contradicts the constrainsti(1). This completes the proof.

From the $FJ-d-\alpha_{\beta}-\rho_{\tau}-\theta-$ pseudoquasi-invex-II and following the previous proofs, we can also establish another weak duality.

Theorem 3.4. (Weak Duality) Let x and (u,λ,μ) be feasible solutions for (MP) and (MD), respectively. If (f,g) is $FJ-d-\alpha_{\beta}-\rho_{\tau}-\theta-$ pseudoquasi-invex-II at u with respect to η, and

$$\frac{\lambda^{T}\rho}{\alpha(x,u)}+\frac{\mu_{J(u)}^{T}\tau_{J(u)}}{\beta(x,u)}\geqq 0$$

With $\lambda>0,\lambda\in R^{k}$. Then the following can not hold:

$$f(x)\leq f(u)$$

The weak dual results allow us to prove the strong duality, as follows

$$f(x)\leq f(u).$$

Theorem 3.5. (Strong Duality) Let $\bar{x}$ be a weakly efficient solution of (MP). If there exist $\bar{\lambda} \in R^k, \bar{\mu} \in R^m$, with $(\bar{\lambda}, \bar{\mu}) \geq 0$, such that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a feasible solution for (MD). Moreover, the assumptions in theorem 3.1 are satisfied, then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a weakly efficient solution of (MD).

Proof: Since $\bar{x}$ be a weakly efficient solution of (MP) and $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a feasible solution of (MD), with the assumptions of theorem 3.1, we can obtain the result

$$f(x^*) < f(\bar{x}),\ \forall (x^*, \lambda^*, \mu^*) \in W, (x^*, \lambda^*, \mu^*) \neq (\bar{x}, \bar{\lambda}, \bar{\mu})$$

Can not hold. It follows that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ a wealy efficient solution of (MD).

Similarly, we can establish the following strong duality.

Theorem 3.6 (Strong Duality) Let $\bar{x}$ be an efficient solution of (MP). If there exist $\bar{\lambda} \in R^k, \bar{\mu} \in R^m$, with $(\bar{\lambda}, \bar{\mu}) \geq 0$, such that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a feasible solution for (MD). Moreover, the assumptions in theorem 3.2 are satisfied, and then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is an efficient solution of (MD).

Theorem 3.7. (Converse Duality) Let $(\bar{u}, \bar{\lambda}, \bar{\mu})$ be a weakly efficient solution for (MD), with $\bar{u}$ feasible for (MP). If (f, g) is $FJ - d - \alpha_\beta - \rho_\tau - \theta -$ pseudoinvex-I at $\bar{u}$ with respect to η, and

$$\frac{\bar{\lambda}^T \rho}{\alpha(\bar{x}, \bar{u})} + \frac{\bar{\mu}^T_{J(\bar{u})} \tau_{J(\bar{u})}}{\beta(\bar{x}, \bar{u})} \geqq 0 \tag{11}$$

Then $\bar{u}$ is a weakly efficient solution for (MP).

Proof: we proceed by contradiction. Suppose that $\bar{u}$ is not a weakly efficient solution for (MP), that is, there exists another $\bar{x} \in D$, such that

$$f(\bar{x}) < f(\bar{u})$$

From the $FJ - d - \alpha_\beta - \rho_\tau - \theta -$ pseudoinvex-I at $\bar{u}$ of (f, g), this yields

$$\alpha(\bar{x}, \bar{u}) f'(\bar{u}; \eta(\bar{x}, \bar{u})) + \rho \left\|\theta(\bar{x}, \bar{u})\right\|^2 < 0$$

$$\beta(\bar{x}, \bar{u}) g'_{J(\bar{u})}(\bar{u}; \eta(\bar{x}, \bar{u})) + \tau_{J(\bar{u})} \left\|\theta(\bar{x}, \bar{u})\right\|^2 < 0$$

That is

$$\alpha(\bar{x}, \bar{u}) f_i'(\bar{u}; \eta(\bar{x}, \bar{u})) + \rho_i \left\|\theta(\bar{x}, \bar{u})\right\|^2 < 0,\ \forall i \in I$$

$$\beta(\bar{x}, \bar{u}) g_j'(\bar{u}; \eta(\bar{x}, \bar{u})) + \tau_j \left\|\theta(\bar{x}, \bar{u})\right\|^2 \leqq 0,\ j \in J(\bar{u})$$

By the feasibility of $(\bar{u}, \bar{\lambda}, \bar{\mu})$ for (MD) and according for the constraints (2)-(3), we have $(\bar{\lambda}, \bar{\mu}_{J(\bar{u})}) \geq 0, \bar{\mu}_{\bar{J}(\bar{u})} = 0$. So, with $\alpha(\bar{x}, \bar{u}) > 0, \beta(\bar{x}, \bar{u}) > 0$, and assumption (11), the above two inequalities imply

$$\begin{aligned}
&\sum_{i \in I} \bar{\lambda}_i f_i'(\bar{u}; \eta(\bar{x}, \bar{u})) + \sum_{j \in J(\bar{u})} \bar{\mu}_j g_j'(\bar{u}; \eta(\bar{x}, \bar{u})) + \sum_{j \in \bar{J}(\bar{u})} \bar{\mu}_j g_j'(\bar{u}; \eta(\bar{x}, \bar{u})) \\
&< -\left(\frac{\sum_{i \in I} \bar{\lambda}_i \rho_i}{\alpha(\bar{x}, \bar{u})} + \frac{\sum_{j \in J(\bar{u})} \bar{\mu}_j \tau_j}{\beta(\bar{x}, \bar{u})} \right) \left\|\theta(\bar{x}, \bar{u})\right\|^2 \\
&= -\left(\frac{\bar{\lambda}^T \rho}{\alpha(\bar{x}, \bar{u})} + \frac{\bar{\mu}^T_{J(\bar{u})} \tau_{J(\bar{u})}}{\beta(\bar{x}, \bar{u})} \right) \left\|\theta(\bar{x}, \bar{u})\right\|^2 \leqq 0
\end{aligned}$$

That is

$$\bar{\lambda}^T f'(\bar{u}; \eta(\bar{x}, \bar{u})) + \bar{\mu}^T g'(\bar{u}; \eta(\bar{x}, \bar{u})) < 0$$

This contradicts the dual constrainst (1). This completes the proof.

In the similar way, we can prove the following converse duality.

Theorem 3.8. (Converse Duality) Let $(\bar{u}, \bar{\lambda}, \bar{\mu})$ be a weakly efficient solution for (MD), with $\bar{u}$ feasible for (MP). If (f, g) is $FJ - d - \alpha_\beta - \rho_\tau - \theta -$ pseudoinvex-II at $\bar{u}$ with respect to η , and

$$\frac{\bar{\lambda}^T \rho}{\alpha(\bar{x}, \bar{u})} + \frac{\bar{\mu}^T_{J(\bar{u})} \tau_{J(\bar{u})}}{\beta(\bar{x}, \bar{u})} \geqq 0$$

Then $\bar{u}$ is an efficient solution for (MP).

4. Discussion and Conclusion

In this paper, we study the multiobjective programming problems and the dual models. Then the weak dual, strong dual and converse dual results are obtained and proved under a class of generalized invexity assumptions the multiobjective programming. The results should be further opportunities for exploiting this structure of the multiobjective programming problems.

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