

Exact Solution of Klein Gordon Equation via Homotopy Perturbation Sumudu Transform Method

Sundas Rubab¹, Jamshaid Ahmad^{*1} and Muhammad Naeem²

¹Department of Mathematics, Faculty of Sciences, University of Gujrat Pakistan

²Department of Mathematics, Faculty of Sciences, University of Engineering & Technology Lahore Pakistan

jamshadahmadm@gmail.com (Jamshad Ahmad)

Abstract

We apply the proposed method (NHPSTM) which is the combination of new homotopy perturbation method and Sumudu transform to solve analytical linear and nonlinear Klein-Gordon equations. The proposed method finds the solution without any discretization or restrictive assumptions and avoids the round-off errors. The fact that the proposed technique solves nonlinear problems without using Adomian's polynomials can be considered as a clear advantage of this new method over the decomposition method. Obtained results reveal that the proposed method is very efficient, simple and can be applied to other nonlinear problems arising in mathematical physics and engineering.

Keywords: Klein-Gordon equation, New homotopy perturbation method, Sumudu transform

1. Introduction

Nonlinear phenomena have important effects on applied mathematics, physics and related to engineering; many such physical phenomena are modeled in terms of nonlinear partial differential equations. The importance of obtaining the exact or approximate solutions of nonlinear partial differential equations in mathematics, physics and engineering is still a significant problem that needs new methods to discover exact or approximate solutions. Various powerful mathematical methods such as variational iteration method [1], homotopy perturbation method [2], new iterative method [3] and Adomian decomposition method [4] have been proven useful for solving algebraic, differential, integro-differential, differential-delay and partial differential equations.

The homotopy perturbation method (HPM) introduced by He [5, 6] has been widely used for solving various integral equations arising from real world modeling, for example thin film flow, heat transfer, and many others [7–19]. The idea behind this method is that the solution is considered as the sum of an infinite series, which converges rapidly to the exact solution. In this paper we present a modified version of HPM and known as NHPM, which performs much better than the HPM. In this paper we outline a reliable strategy of the new Homotopy Perturbation Sumudu Transform Method (HPSTM) for solving the Klein Gordon equation

$$u_{tt} + \alpha u_{xx} + \beta u + \gamma u^k = f(x, t), \quad (1)$$

with the initial conditions

$$u(x, 0) = g_1(x), \quad u_t(x, 0) = g_2(x), \quad (2)$$

Where α, β and γ are known constants, when $k = 2$ we have quadratic nonlinearity and when $k = 3$ we have cubic nonlinearity; $f(x, t)$, $g_1(x)$ and $g_2(x)$ are known functions, and the function $u(x, t)$ is unknown.

2. Definitions and Properties of the S–Transform

Watuğala [23] introduced a new integral transform, named the Sumudu transform and applied it to the solution of ordinary differential equation in control engineering problems. Sumudu transform is defined over the set of the following functions

$$A = \{ f(t) : \tau_1, \tau_2 > 0, |f(t)| < Me^{t/\tau_j}, \text{ if } t \in (-1)^j \times [0, \infty) \} \quad (3)$$

By the following formula

$$G(u) = S[f(t); u] := \int_0^\infty f(ut) e^{-t} dt, \quad u \in (-\tau_1, \tau_2). \quad (4)$$

Sumudu transforms of the derivatives of (x) is

$$S\left[\frac{d^n U}{dx^n}\right] = \frac{1}{u^n} S[U(x)] - \frac{1}{u^n} U(0) - \frac{1}{u^{n-1}} U'(0) - \dots - \frac{U^{n-1}(0)}{u}. \quad (5)$$

Some properties of the Sumudu transform are as follows

1. $S[1] = 1$.
2. $S\left[\frac{t^n}{\Gamma(n+1)}\right] = u^n, n > 0$.
3. $S[e^{at}] = \frac{1}{1-au}$.
4. $S[\alpha f(x) + \beta g(x)] = \alpha S[f(x)] + \beta S[g(x)]$.

Other properties of the Sumudu transform can be found in [22].

3. Analysis of NHPSM to Klein–Gordon Equations

To solve Eq. (1) by NHPM we construct the following homotopy

$$\frac{\partial^2 U}{\partial t^2} = u_0 - p \left(u_0 + \alpha \frac{\partial^2 U}{\partial x^2} + \beta U + \gamma U^k - f(x, t) \right). \quad (6)$$

By applying the Sumudu transform, we obtain

$$S\left[\frac{\partial^2 U}{\partial t^2}\right] = S\left[u_0 - p \left(u_0 + \alpha \frac{\partial^2 U}{\partial x^2} + \beta U + \gamma U^k - f(x, t) \right)\right]. \quad (7)$$

then

$$S[U(x, t)] = U(x, 0) + uU_t(x, 0) + u^2 S\left[u_0 - p \left(u_0 + \alpha \frac{\partial^2 U}{\partial x^2} + \beta U + \gamma U^k - f(x, t) \right)\right], \quad (8)$$

where $U(x, 0) = g_1(x)$ and $U_t(x, 0) = g_2(x)$.

Applying inverse Sumudu transform on both sides

$$U(x, t) = S^{-1}\left[U(x, 0) + uU_t(x, 0) + u^2 S\left[u_0 - p \left(u_0 + \alpha \frac{\partial^2 U}{\partial x^2} + \beta U + \gamma U^k - f(x, t) \right)\right]\right].$$

(9)

Assume the solution of Eq. (9) to have the form

$$U = U_0 + pU_1 + p^2U_2 + \dots \quad (10)$$

where U_i are unknown functions to be determined.

Now suppose that the initial approximation to the solution $u_0(x, t)$ has the form

$$u_0(x, t) = \sum_{n=0}^{\infty} a_n(x)P_n(t), \quad (11)$$

where $a_1(t), a_2(t), a_3(t), \dots$ are unknown coefficients and $P_0(x), P_1(x), P_2(x), \dots$ are specific functions depending on the problem. Substituting Eq. (10) into Eq. (9), collecting the same powers of p , and equating each coefficient of p , we obtain

$$\begin{aligned} p^0; U_0(x, t) &= S^{-1} [U(x, 0) + uU_1(x, 0) + u^2 S(u_0)], \\ p^1; U_1(x, t) &= -S^{-1} \left[u^2 \left(u_0 + \alpha \frac{\partial^2 U_0}{\partial x^2} + \beta U_0 + \gamma U_0^k - f(x, t) \right) \right], \\ p^2; U_2(x, t) &= -S^{-1} \left[u^2 \left(\alpha \frac{\partial^2 U_1}{\partial x^2} + \beta U_1 + \gamma U_1 U_0^{k-1} \right) \right], \\ &\vdots \end{aligned} \quad (12)$$

Now if we solve these equations in such a way that $U_1(x, t) = U_2(x, t) = \dots = 0$. Therefore the exact solution may be obtained as

$$u(x, t) = U_0(x, t) = \sum_{n=0}^{\infty} a_n(x)P_n(t).$$

If $f(x, t)$ and $u_0(x, t)$ are analytic at $t = t_0$. then their Taylor series defined as

$$\begin{aligned} u_0(x, t) &= \sum_{n=0}^{\infty} a_n(x)(t - t_0)^n, \\ f(x, t) &= \sum_{n=0}^{\infty} a_n^*(x)(t - t_0)^n. \end{aligned}$$

can be used in Eq. (12), where $a_1(x), a_2(x), a_3(x), \dots$ are unknown coefficients and $a_1^*(x), a_2^*(x), a_3^*(x), \dots$ are known ones, which must be computed.

4. Numerical Applications

Example 4.1 Consider the linear Klein–Gordon equation [20]

$$u_{tt} - u_{xx} - 2u = -2 \sin x \sin t, \quad (13)$$

with initial conditions

$$\begin{aligned} u(x, 0) &= 0, \\ \frac{\partial u(x, 0)}{\partial t} &= \sin x. \end{aligned}$$

To solve Eq. (13) by the NHPM, we construct the following homotopy

$$\frac{\partial^2 U}{\partial t^2} = u_0 - p \left(u_0 - \frac{\partial^2 U}{\partial x^2} - 2U + 2 \sin x \sin t \right), \quad (14)$$

Applying the Sumudu transform on both sides of eq. (14)

$$S \left[\frac{\partial^2 U}{\partial t^2} \right] = S \left[u_0 - p \left(u_0 - \frac{\partial^2 U}{\partial x^2} - 2U + 2 \sin x \sin t \right) \right],$$

$$S[U(x, t)] = u \sin x + u^2 S \left[u_0 - p \left(u_0 - \frac{\partial^2 U}{\partial x^2} - 2U + 2 \sin x \sin t \right) \right], \quad (15)$$

By taking inverse Sumudu transform

$$U(x, t) = t \sin x + S^{-1} \left[u^2 S \left(u_0 - p \left(u_0 - \frac{\partial^2 U}{\partial x^2} - 2U + 2 \sin x \sin t \right) \right) \right], \quad (16)$$

Assume the solution of Eq. (16) to have the following form

$$U = U_0 + pU_1 + p^2U_2 + \dots \quad (17)$$

where U_i are unknown functions to be determined. Substituting Eq. (17) into Eq. (16), collecting the same powers of p , and equating each coefficient of p , we obtain

$$p^0; U_0(x, t) = t \sin x + S^{-1} [u^2 S(u_0)],$$

$$p^1; U_1(x, t) = -S^{-1} \left[u^2 S \left(u_0 - \frac{\partial^2 U_0}{\partial x^2} - 2U_0 + 2 \sin x \sin t \right) \right],$$

$$p^2; U_2(x, t) = S^{-1} \left[u^2 S \left(\frac{\partial^2 U_1}{\partial x^2} + 2U_1 \right) \right],$$

$$\vdots$$

$$p^{j+1}; U_{j+1}(x, t) = S^{-1} \left[u^2 S \left(\frac{\partial^2 U_j}{\partial x^2} + 2U_j \right) \right],$$

$$\vdots,$$

Assuming

$$u_0(x, t) = \sum_{n=0}^{\infty} a_n(x)t^n, \quad U(x, 0) = u(x, 0), \quad U_t(x, 0) = u_t(x, 0)$$

and solving the above equation $U_1(x, t)$ for we have

$$U_1(x, t) = \left(-\frac{1}{2} a_0(x) \right) t^2 + \left(-\frac{1}{3!} a_1(x) + \frac{1}{3!} \sin x \right) t^3 + \left(-\frac{1}{12} a_2(x) + \frac{1}{24} a_{0,xx}(x) + \frac{1}{12} a_0(x) \right) t^4 + \dots$$

we obtain

$$a_0(x) = 0, \quad a_1(x) = -\sin x, \quad a_2(x) = 0, \dots$$

Therefore, the exact solution is

$$u(x, t) = U_0(x, t) = t \sin x + \frac{1}{2} a_0(x)t^2 + \frac{1}{3!} a_1(x)t^3 + \dots = \sin x \sin t. \quad (18)$$

Example4.2 Consider the nonlinear inhomogeneous Klein–Gordon equation [21]

$$u_{tt} - u_{xx} + u^2 = -x \cos t + x^2 \cos^2 t, \quad (19)$$

with initial conditions

$$u(x,0) = x,$$

$$\frac{\partial u(x,0)}{\partial t} = 0.$$

To solve Eq. (19) by the NHPM, we construct the following homotopy

$$\frac{\partial^2 U}{\partial t^2} = u_0 - p \left(u_0 - \frac{\partial^2 U}{\partial x^2} + U^2 + x \cos t - x^2 \cos^2 t \right), \quad (20)$$

Taking Sumudu transform on both sides,

$$S \left[\frac{\partial^2 U}{\partial t^2} \right] = S \left[u_0 - p \left(u_0 - \frac{\partial^2 U}{\partial x^2} + U^2 + x \cos t - x^2 \cos^2 t \right) \right],$$

Then

$$S[U(x,t)] = x + u^2 S \left[u_0 - p \left(u_0 - \frac{\partial^2 U}{\partial x^2} + U^2 + x \cos t - x^2 \cos^2 t \right) \right], \quad (21)$$

Applying inverse Sumudu transform

$$U(x,t) = x + S^{-1} \left[u^2 S \left(u_0 - p \left(u_0 - \frac{\partial^2 U}{\partial x^2} + U^2 + x \cos t - x^2 \cos^2 t \right) \right) \right], \quad (22)$$

Assume the solution of Eq. (22) to have the following form

$$U = U_0 + pU_1 + p^2U_2 + \dots \quad (23)$$

substituting Eq. (23) into Eq. (22), collecting terms with the same powers of p , and equating each coefficient of p to zero, we obtain

$$p^0; U_0(x,t) = x + S^{-1} [u^2 S(u_0)],$$

$$p^1; U_1(x,t) = -S^{-1} \left[u^2 S \left(u_0 - \frac{\partial^2 U_0}{\partial x^2} + U_0^2 + x \cos t - x^2 \cos^2 t \right) \right],$$

$$p^2; U_2(x,t) = S^{-1} \left[u^2 S \left(\frac{\partial^2 U_1}{\partial x^2} - 2U_0U_1 \right) \right],$$

⋮

$$p^{j+1}; U_{j+1}(x,t) = S^{-1} \left[u^2 S \left(\frac{\partial^2 U_j}{\partial x^2} - \sum_{k=0}^j U_j U_{j-k} \right) \right],$$

⋮.

and

$$u_0(x,t) = \sum_{n=0}^{\infty} a_n(x)t^n, \quad U(x,0) = u(x,0), \quad U_t(x,0) = u_t(x,0)$$

and solving the above equation $U_1(x,t)$ for we have

$$U_1(x,t) = \left(-\frac{1}{2}a_0(x) - \frac{1}{2}x \right)t^2 + \left(-\frac{1}{3!}a_1(x) \right)t^3 + \left(-\frac{1}{12}a_2(x) - \frac{1}{24}a_{0,xx}(x) - \frac{1}{12}xa_0(x) + \frac{1}{24}x - \frac{1}{6}x^2 \right)t^4 + \dots = 0,$$

$$a_0(x) = -x, \quad a_1(x) = 0, \quad a_2(x) = \frac{1}{2}x, \dots$$

Therefore the exact solution is

$$u(x, t) = U_0(x, t) = x + \frac{1}{2} a_0(x)t^2 + \frac{1}{3!} a_1(x)t^3 + \frac{1}{12} a_2(x)t^4 + \dots = x \cos t. \quad (24)$$

Example4.3 Finally, we consider the nonlinear Klein–Gordon equation with cubic nonlinearity

$$u_{tt} + u_{xx} + u + u^3 = 2x + xt^2 + x^3t^6, \quad (25)$$

with initial conditions

$$u(x, 0) = 0,$$

$$\frac{\partial u(x, 0)}{\partial t} = 0.$$

To solve Eq. (25) by the NHPM, we consider the following homotopy

$$\frac{\partial^2 U}{\partial t^2} = u_0 - p \left(u_0 + \frac{\partial^2 U}{\partial x^2} + U + U^3 - 2x - xt^2 - x^3t^6 \right), \quad (26)$$

Applying Sumudu transform on both sides,

$$S \left[\frac{\partial^2 U}{\partial t^2} \right] = S \left[u_0 - p \left(u_0 + \frac{\partial^2 U}{\partial x^2} + U + U^3 - 2x - xt^2 - x^3t^6 \right) \right], \quad (27)$$

Then

$$S[U(x, t)] = u^2 S \left[u_0 - p \left(u_0 + \frac{\partial^2 U}{\partial x^2} + U + U^3 - 2x - xt^2 - x^3t^6 \right) \right], \quad (28)$$

Applying inverse Sumudu transform

$$U(x, t) = S^{-1} \left[u^2 S \left(u_0 - p \left(u_0 + \frac{\partial^2 U}{\partial x^2} + U + U^3 - 2x - xt^2 - x^3t^6 \right) \right) \right], \quad (29)$$

Assume the solution of Eq. (29) as

$$U = U_0 + pU_1 + p^2U_2 + \dots \quad (30)$$

substituting Eq. (30) into Eq. (29), collecting terms with the same powers of p , and equating each coefficient of p to zero, we obtain

$$p^0; U_0(x, t) = S^{-1} \left[u^2 S(u_0) \right],$$

$$p^1; U_1(x, t) = -S^{-1} \left[u^2 S \left(u_0 + \frac{\partial^2 U_0}{\partial x^2} + U_0 + U_0^3 - 2x - xt^2 - x^3t^6 \right) \right],$$

$$p^2; U_2(x, t) = -S^{-1} \left[u^2 S \left(\frac{\partial^2 U_1}{\partial x^2} + U_1 + 3U_0U_1^2 \right) \right],$$

⋮

$$p^{j+1}; U_{j+1}(x, t) = -S^{-1} \left[u^2 S \left(\frac{\partial^2 U_j}{\partial x^2} + U_j + \sum_{i=0}^j \sum_{k=0}^{j-i} U_i U_k U_{j-k-i} \right) \right],$$

⋮

and

$$u_0(x, t) = \sum_{n=0}^{\infty} a_n(x)t^n, \quad U(x, 0) = u(x, 0), \quad U_t(x, 0) = u_t(x, 0)$$

and solving the above equation $U_1(x, t)$ for we have

$$U_1(x, t) = \left(-\frac{1}{2}a_0(x) + x \right)t^2 + \left(-\frac{1}{3!}a_1(x) \right)t^3 + \left(-\frac{1}{12}a_2(x) - \frac{1}{24}a_{0,xx}(x) - \frac{1}{24}a_0(x) + \frac{1}{12}x \right)t^4 + \dots$$

Eliminating $U_1(x, t)$ makes the coefficients $a_n(x)(n = 1, 2, 3, \dots)$ take the following values:

$$a_0(x) = 2x, \quad a_1(x) = a_2(x) = a_3(x) = \dots = 0.$$

Therefore the exact solution is

$$u(x, t) = U_0(x, t) = \frac{1}{2}a_0(x)t^2 + \frac{1}{3!}a_1(x)t^3 + \frac{1}{12}a_2(x)t^4 + \dots = xt^2. \quad (31)$$

5. Conclusions

In this paper, we have successfully applied the new homotopy perturbation Sumudu transform method (NHPSTM) for finding exact solutions of linear and nonlinear Klein-Gordon equations. Numerical results show that proposed method is a powerful mathematical tool for solving the Klein-Gordon equation. Therefore, it may be concluded that the proposed method is a promising technique in finding exact solutions for a wide variety of mathematical problems arising in science and engineering.

References

- [1] J. H. He, "Variational iteration method-akind of nonlinear analytical technique: some examples", *International Journal of Nonlinear Mechanics*, vol. 34, (1999), pp. 699-708.
- [2] J. H. He, "Homotopy perturbation technique", *Computer Methods in Applied Mechanics and Engineering*, vol. 178, (1999), pp. 257-262.
- [3] V. Daftardar-Gejji and H. Jafari, "An iterative method for solving nonlinear functional equations", *Journal of Mathematical Analysis and Applications*, vol. 316, no. 2, (2006), pp. 753-763.
- [4] G. Adomian, "Solving frontier problems of physics: The decomposition method", *Kluwer Acad. Publ.*, (1994).
- [5] J. H. He, Homotopy perturbation technique, *Comput. Methods in Appl. Mech. Eng.*, vol. 178, (1999), pp. 257-262.
- [6] J. H. He, "A coupling method of homotopy technique and perturbation technique for nonlinear problems", *Int. J. Nonlinear Mech.*, vol. 35, no. 1, (2000), pp. 37-43.
- [7] M. Dehghan and F. Shakeri, "Use of He's homotopy perturbation method for solving a partial differential equation arising in modeling of flow in porous media", *J. Porous Media*, vol. 11, (2008), pp. 765-778.
- [8] J. Biazar, H. Ghazvini, and M. Eslami, "He's homotopy perturbation method for systems of integro-differential equations", *Chaos, Solitons & Fractals*, vol. 39, (2009), pp. 1253-1258.
- [9] J. Ahmad and S. T. Mohyud-Din, "Solving fractional vibrational problem using restarted fractional Adomian's decomposition method", *Life Sci J*; vol. 10, no. 4, (2013), pp. 210-216.
- [10] D. D. Ganji, "The application of He's homotopy perturbation method to nonlinear equations arising in heat transfer", *Phys. Lett. A*, vol. 355, (2006), pp. 337-341.
- [11] L. Cveticanin, "Homotopy-perturbation method for pure nonlinear differential equation", *Chaos, Solitons & Fractals*, vol. 30, (2006), pp. 1221-1230.
- [12] A. Beléndez, T. Beléndez, A. Márquez, and C. Neipp, "Application of He's homotopy perturbation method to conservative truly nonlinear oscillators", *Chaos, Solitons & Fractals*, vol. 37, no. 3, (2008), pp. 770-780.
- [13] S. Abbasbandy, "Numerical solutions of the integral equations: Homotopy perturbation method and Adomian's decomposition method", *Appl. Math. Comput.*, vol. 173, (2006), pp. 493-500.
- [14] A. Golbabai and M. Javidi, "Application of homotopy perturbation method for solving eighth-order boundary value problems", *Appl. Math. Comput.*, vol. 191, no. 1, (2007), pp. 334-346.

- [15] M. A. Rana, A. M. Siddiqui, and Q. K. Ghori, "Application of He's homotopy perturbation method to Sumudu transform", *Int. J. Nonlinear Sci. Numer. Simul.*, vol. 8, no. 2, (2007), pp. 185–190.
- [16] T. Ozis and A. Yildirim, "Traveling wave solution of Korteweg–de Vries equation using He's homotopy perturbation method", *Int. J. Nonlinear Sci. Numer. Simul.*, vol. 8, no. 2, (2007), pp. 239–242.
- [17] J. Biazar, M. Eslami, and H. Ghazvini, "Homotopy perturbation method for systems of partial differential equations", *Int. J. Nonlinear Sci. Numer. Simul.*, vol. 8, no. 3, (2007), pp. 411–416.
- [18] A. Yildirim, "Application of He's homotopy perturbation method for solving the Cauchy reaction–diffusion problem", *Comput. Math. Appl.*, vol. 57, no. 4, (2009), pp. 612–618.
- [19] M. A. Noor and S. Tauseef Mohyud-Din, "Homotopy perturbation method for solving sixth-order boundary value problems", *Comput. Math. Appl.*, vol. 55, (2008), pp. 2953–2972.
- [20] E. Y. Deeba and S. A. Khuri, "A decomposition method for solving the nonlinear Klein–Gordon equation", *J. Comput. Phys.*, vol. 124, (1996), pp. 442–448.
- [21] A. M. Wazwaz, "The modified decomposition method for analytic treatment of differential equations", *Appl. Math. Comput.*, vol. 173, (2006), pp. 165–176.
- [22] F. B. M. Belgacem and A. A. Karaballi, "Sumudu transform fundamental properties investigations and applications", *International J. Appl. Math. Stoch. Anal.*, V vol. 2006 DOI 10.1155/JAMSA/2006/91083, 2005, pp. 1–23.
- [23] G. K. Watugala, "Sumudu transform- a new integral transform to solve differential equations and control engineering problems", *Math. Engg. Indust.*, vol. 6, no. 4, (1998), pp. 319–329.