

Robust Stabilization of a Class of Uncertain Fractional-order Chaotic Systems via a Novel Sliding Mode Control Scheme

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Abstract

This paper proposes a novel sliding mode control (SMC) scheme to stabilize a class of fractional-order chaotic systems. Through constructing two sliding mode variables, the control problem of n -dimensional system can be transformed to the equivalent stabilizing problem of a reduced-order system. Subsequently, on the basis of second-order sliding mode (SOSM) technique, a robust control law is designed, which strongly attenuates the chattering phenomenon inherent in traditional sliding mode controller, and guarantees the existence of sliding motion in a finite time. The stability of two sliding mode variables to the origin is proved by conventional and fractional Lyapunov theories, respectively. Finally, two numerical examples are provided to illustrate the effectiveness of the proposed approach.

Keywords: *Fractional-order chaotic system; Second-order sliding mode control; Fractional Lyapunov theory; Robust stabilization*

1. Introduction

Although fractional calculus is a mathematical topic with more than 300 years history, its application in the fields of physics and engineering has attracted lots of attentions only in the recent years. It was found that, with the help of fractional calculus, many systems in interdisciplinary fields can be described more accurately, such as viscoelastic system [1], dielectric polarization [2], electrode-electrolyte polarization [3], finance systems and electromagnetic waves [4]. That is to say, fractional derivatives provide a superb instrument for the description of memory and hereditary properties of various materials and processes. Meanwhile, many literatures have proven that some fractional-order differential systems can behave chaotically, *e.g.*, fractional-order Duffing system [5], fractional-order Chen-Lee system [6], fractional-order Lorenz system [7], fractional-order hyperchaotic Chen system [8], fractional-order Qi system [9], and so on.

Recently, studying fractional-order systems has become an active area. In particular, control and synchronization of fractional-order chaotic systems have attracted much more attentions from various scientific fields. Some methods have been proposed to achieve chaos control/synchronization in fractional-order chaotic systems. For example, Odibat [10] proposed the nonlinear feedback control scheme to realize the synchronization of two non-identical fractional-order chaotic systems; Lu [11] introduced the pole placement technique to design an observer for control fractional-order chaotic systems; Targhvafard *et al.* [12] developed the active control method to proceed phase and anti-phase synchronization of fractional-order chaotic systems. Due to the useful properties of sliding mode control (SMC), such as low sensitivity to parameters disturbances, robustness to the systems' uncertainties, fast convergence and easy design in practice, SMC approach has become a universal method

to realize the control/synchronization of fractional-order chaotic systems in the past decades [13-17].

However, the traditional SMC is of the first order, and there exists an inevitable drawback in applying such SMC, that is the so-called chattering phenomenon cannot be avoided. In order to improve the control accuracy and avoid, or at least strongly attenuate the undesired chattering effect, the second-order sliding mode (SOSM) method is proposed. The SOSM control approach was developed starting from the mid eighties, and its main aim is enhancing the control precision and eliminating the chattering phenomenon by removing the control discontinuity. To the best of our knowledge, so far, there are few works available in stabilizing/synchronizing fractional-order chaotic systems by SOSM control approach.

Take the above discussions into account, this article considers the robust stabilization problem of a class of fractional-order chaotic systems in the presence of model uncertainties and external disturbances. The main contribution of this article is that using the novel SMC approach, the control task of n-dimensional system is transformed to the equivalent problem of stabilizing a reduced-order system. In other words, under the effect of the proposed method, if the reduced-order system is stable, then the n-dimensional system also is stable, so we only need to ensure the asymptotic stability of the reduced-order system. We use the traditional and fractional Lyapunov functions to demonstrate that the reaching phases of two sliding surfaces are stable. Besides, on the basis of SOSM control theory, a robust control law is designed, which can strongly attenuates the chattering phenomenon inherent in conventional sliding mode controller.

The structure of this paper is organized as follows. In Section 2, relevant definitions and lemmas are presented. Section 3 gives the main results. In Section 4, two simulation examples are provided to verify the effectiveness of the proposed method. Conclusions are drawn in Section 5.

2. Preliminaries

The Riemann-Liouville and Caputo definitions are the frequently used definitions of fractional calculus.

Definition 1 The α order Riemann-Liouville fractional integration is given by

$${}_{t_0} I_t^\alpha = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \quad (1)$$

where $\Gamma(\alpha)$ is the Gamma function, determined by

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt \quad (2)$$

Definition 2 For $n-1 < \alpha \leq n$, $n \in R$, the α order Riemann-Liouville fractional derivative is defined as

$${}_{t_0} D_t^\alpha f(t) = \frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau = \frac{d^n}{dt^n} I^{n-\alpha} f(t) \quad (3)$$

Definition 3 The α order Caputo fractional derivative is written as

$${}_{t_0} D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, & m-1 < \alpha < m \\ \frac{d^m}{dt^m} f(t), & \alpha = m \end{cases} \quad (4)$$

where m is the smallest integer number, larger than α .

Lemma 1 (see [18]) Consider a system $D^\alpha x(t) = Ax(t)$, the system is asymptotically stable if and only if $|\arg(\text{spec}(A))| > \alpha\pi/2$, in this case, each state of system converge to zero like $t^{-\alpha}$.

Lemma 2 (see [19]) Consider the system

$$\dot{x}(t) = f(x(t)), \quad f(0) = 0, \quad x(t) \in R^n \quad (5)$$

where $f : D \rightarrow R^n$ is continuous on an open neighborhood $D \subset R^n$. Suppose there exists a continuous differential positive definite function $V(x(t)) : D \rightarrow R$, real numbers $p > 0$, $0 < \eta < 1$, such that

$$\dot{V}(x(t)) + pV^\eta(x(t)) \leq 0, \quad \forall x(t) \in D$$

Then the origin of system (5) is a locally finite time stable equilibrium, and the settling time, depending on the initial state $x(0) = x_0$, satisfies $T(x_0) = V^{1-\eta}(x_0)/p(1-\eta)$. In addition, if $D = R^n$ and $V(x(t))$ is also radially unbounded, then the origin is a globally finite time stable equilibrium of system (5).

Lemma 3 (see [20]) If $\alpha < 2$, β is an arbitrary real number, μ is such that $\alpha\pi/2 < \mu < \min\{\pi, \alpha\pi\}$, and K is a real positive constant, then

$$|E_{\alpha,\beta}(z)| \leq \frac{K}{1+|z|}, \quad (\mu \leq |\arg(z)| \leq \pi), \quad |z| > 0$$

Lemma 4 (see [20]) Consider a two-parameters Mittag-Leffler function, the Laplace transform satisfying

$$\mathcal{L}\{t^{\beta-1}E_{\alpha,\beta}(-\lambda t^\alpha)\} = \frac{s^{\alpha-\beta}}{s^\alpha + \lambda}, \quad (R(s) > |\lambda|^{1/\alpha})$$

where t and s are the variables in time domain and Laplace domain, respectively, $\mathcal{L}\{\cdot\}$ stands for the Laplace transform.

3. Main Results

3.1. Design of Chattering-free SOSM Controller for Uncertain Fractional-order Chain of Integrator

Consider the following n-dimensional uncertain fractional-order chain of integrator

$$\begin{aligned} D^\alpha x_1 &= x_2 \\ D^\alpha x_2 &= x_3 \\ &\vdots \\ D^\alpha x_n &= f(x) + \Delta f(x) + d(t) + u(t) \end{aligned} \quad (6)$$

where $\alpha \in (0,1)$ is the fractional order of the system, $x(t) = [x_1, x_2, \dots, x_n]^T$ is the state vector, $f(x) : R^n \rightarrow R$ is a given nonlinear function of x , $\Delta f(x)$ and $d(t)$ represent model uncertainty and external disturbance, respectively, $u(t)$ is the single control input to be designed later.

Assumption 1. The uncertainty term $\Delta f(x)$ and external disturbance $d(t)$ are derivable, the bounds of their derivatives is a known positive constant δ :

$$\left| \frac{d}{dt}(\Delta f(x) + d(t)) \right| \leq \delta$$

Remark 1. In order to design a chattering-free SOSM controller, the smoothness hypotheses of the uncertainty term and external disturbance are required as in Assumption 1, which is not necessary with the first-order SMC approach. Therefore, this can be seen as a standard assumption in applying SOSM technique.

Our control task is to design a chattering-free SOSM controller for stabilization of the fractional-order system (6) around zero asymptotically.

In this section, we firstly establish a simple linear sliding mode surface to convert the control problem of the n-dimensional system to the equivalent stabilization problem of a reduced-order system, given by

$$\psi = x_n + \sum_{i=1}^{n-1} c_i x_i \quad (7)$$

where c_i , ($i = 1, 2, \dots, n - 1$) are sliding surface parameters to be designed later. Once the system states operate in sliding mode, the following equations are holds:

$$\psi = 0, \dot{\psi} = 0 \quad (8)$$

Consequently, using (7) we have

$$x_n = -\sum_{i=1}^{n-1} c_i x_i \quad (9)$$

Now, according to (9) we can rewrite the first n-1 equations of (6) as

$$\begin{aligned} D^\alpha x_1 &= x_2 \\ D^\alpha x_2 &= x_3 \\ &\vdots \\ D^\alpha x_{n-1} &= -\sum_{i=1}^{n-1} c_i x_i \end{aligned}$$

or in a matrix equation form as

$$D^\alpha \bar{x} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -c_1 & -c_2 & -c_3 & \dots & -c_{n-1} \end{pmatrix} \bar{x} = A \bar{x} \quad (10)$$

where $\bar{x} = [x_1, x_2, \dots, x_{n-1}]^T$, it is obvious that system (10) form a reduced-order system (as compared to (6)). In order to guarantee the stability of system (10), according to Lemma 1, the sliding surface parameters c_i are selected to be positive such that the eigenvalues of the matrix A have negative real part. Noticing that if system (10) is asymptotically stable, then we can conclude from (9) that x_n is asymptotically stable, too, that is the system (6) is stabilized asymptotically. Thus, under the effect of the sliding mode variable ψ , the control problem of n-dimensional system is transformed to the stabilization problem of the reduced-order system.

In order to eliminate the chattering effect, SOSM technique is used to derive the control law. For getting the desired dynamics (9), we construct another dynamical fractional-order integral type sliding surface, determined by:

$$\sigma = D^{\alpha-1} |\psi| + \eta_1 D^{\alpha-1} \psi + \eta_2 \int_0^t (\psi + \text{sgn}(\psi)) d\tau \quad (11)$$

where $\eta_1 > 1$, and $\eta_2 > 0$, the SOSM controller will be designed to drive σ converge to zero in finite time. Take the time derivative for both sides of (11), we have

$$\dot{\sigma} = D^\alpha |\psi| + \eta_1 D^\alpha \psi + \eta_2 (\psi + \text{sgn}(\psi)) \quad (12)$$

when σ operates in sliding mode, one has

$$\sigma = 0, \dot{\sigma} = 0$$

that is, the following sliding mode dynamics can be obtained

$$D^\alpha \psi = - \frac{\eta_2 (\psi + \text{sgn}(\psi))}{\text{sgn}(\psi) + \eta_1} \quad (13)$$

Next, we will provide several important theorems.

Theorem 1 Under the Assumption 1, consider the uncertain fractional-order system (6) and the sliding mode variables (7) and (11), if the SOSM control law is designed as:

$$\begin{aligned} u(t) &= -f(x) - \sum_{i=1}^{n-1} c_i x_{i+1} - (\text{sgn}(\psi) + \eta_1)^{-1} [\eta_2 (x_n + \sum_{i=1}^{n-1} c_i x_i + \text{sgn}(\psi)) \\ &\quad + k_1 |\sigma|^{1/2} \text{sgn}(\sigma) + k_2 \sigma - z] \\ \dot{z} &= -k_3 \text{sgn}(\sigma) - k_4 \sigma \end{aligned} \quad (14)$$

where sgn is the sign function; $k_1, k_2, k_3, k_4 > 0$ denote the design parameters satisfying

$$\begin{aligned} k_1 &> 0, \quad k_2 > 2\sqrt{\bar{\delta}}, \quad k_3 > \bar{\delta} \\ k_4 &> k_2^2 \frac{k_1^2 + \frac{5}{2} \left(\frac{1}{4} k_1^2 - \bar{\delta} + k_1 k_3 \right)}{\frac{1}{4} k_1^2 - \bar{\delta} + k_1 k_3} \end{aligned} \quad (15)$$

where $\bar{\delta} = (1 + \eta_1)\delta$, then σ will converge to zero in finite time.

Proof According to Definition 2, we have

$$\begin{aligned} \dot{\sigma} &= \frac{d}{dt} \left(D^{\alpha-1} |\psi| + \eta_1 D^{\alpha-1} \psi + \eta_2 \int_0^t (\psi + \text{sgn}(\psi)) d\tau \right) \\ &= D^\alpha |\psi| + \eta_1 D^\alpha \psi + \eta_2 (\psi + \text{sgn}(\psi)) \\ &= (\text{sgn}(\psi) + \eta_1) D^\alpha \psi + \eta_2 (\psi + \text{sgn}(\psi)) \end{aligned} \quad (16)$$

Substituting (7) into the (16), and according to (6), it yields

$$\begin{aligned} \dot{\sigma} &= (\text{sgn}(\psi) + \eta_1) \left(D^\alpha x_n + \sum_{i=1}^{n-1} c_i x_{i+1} \right) + \eta_2 \left(x_n + \sum_{i=1}^{n-1} c_i x_i + \text{sgn}(\psi) \right) \\ &= (\text{sgn}(\psi) + \eta_1) \left(f(x) + \Delta f(x) + d(t) + u(t) + \sum_{i=1}^{n-1} c_i x_{i+1} \right) \\ &\quad + \eta_2 \left(x_n + \sum_{i=1}^{n-1} c_i x_i + \text{sgn}(\psi) \right) \end{aligned} \quad (17)$$

Inserting the first equation of (14) into (17), one has

$$\begin{aligned} \dot{\sigma} &= (\text{sgn}(\psi) + \eta_1) \left[f(x) + \Delta f(x) + d(t) + (-f(x) - \sum_{i=1}^{n-1} c_i x_{i+1} - (\text{sgn}(\psi) + \eta_1)^{-1} \times \right. \\ &\quad \left. (\eta_2 (x_n + \sum_{i=1}^{n-1} c_i x_i + \text{sgn}(\psi)) + k_1 |\sigma|^{1/2} \text{sgn}(\sigma) + k_2 \sigma - z)) + \sum_{i=1}^{n-1} c_i x_{i+1} \right] \\ &\quad + \eta_2 (x_n + \sum_{i=1}^{n-1} c_i x_i + \text{sgn}(\psi)) \\ &= -k_1 |\sigma|^{1/2} - k_2 \sigma + z + \varphi(t) \\ \dot{z} &= -k_3 \text{sgn}(\sigma) - k_4 \sigma \end{aligned} \quad (18)$$

where $\varphi(t) = (\text{sgn}(\psi) + \eta_1) (\Delta f(x) + d(t))$. From Assumption 1, we have

$$\left| \frac{d}{dt} \varphi(t) \right| \leq (1 + \eta_1) \left| \frac{d}{dt} (\Delta f(x) + d(t)) \right| \leq (1 + \eta_1) \delta = \bar{\delta} \quad (19)$$

Letting $\rho = z + \varphi(t)$, then (18) can be rewritten as

$$\begin{aligned} \dot{\sigma} &= -k_1 |\sigma|^{1/2} \text{sgn}(\sigma) - k_2 \sigma + \rho \\ \dot{\rho} &= -k_3 \text{sgn}(\sigma) - k_4 \sigma + \frac{d}{dt} \varphi(t) \end{aligned} \quad (20)$$

Selecting a Lyapunov function for (20)

$$V_1(t) = 2k_3 |\sigma| + k_4 \sigma^2 + \frac{1}{2} \rho^2 + \frac{1}{2} (k_1 |\sigma|^{1/2} \text{sgn}(\sigma) + k_2 \sigma - \rho)^2 \quad (21)$$

which can also be written as a quadratic form $V_1(t) = \xi^T P \xi$, where

$$\xi = [|\sigma|^{1/2} \operatorname{sgn}(\sigma) \quad \sigma \quad \rho]^T$$

$$P = \frac{1}{2} \begin{pmatrix} 4k_3 + k_1^2 & k_1 k_2 & -k_1 \\ k_1 k_2 & 2k_4 + k_2^2 & -k_2 \\ -k_1 & -k_2 & 2 \end{pmatrix}$$

It is obviously that $V_1(t)$ is positive and radially unbounded if $k_3 > 0$, moreover, it satisfies

$$\lambda_{\min}(P) \|\xi\|_2^2 \leq V_1(t) \leq \lambda_{\max}(P) \|\xi\|_2^2$$

where $\|\xi\|_2^2 = |\sigma| + \sigma^2 + \rho^2$ is the Euclidean 2-norm of ξ , $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are the smallest eigenvalue and the largest eigenvalue of matrix P , respectively. Taking the time derivative of $V_1(t)$ along the system (20), and through simple operation, we have

$$\begin{aligned} \dot{V}_1(t) = & -(k_2 k_3 + 2k_1^2 k_2) |\sigma| - (k_2 k_4 + k_2^3) \sigma^2 + 2k_2^2 \sigma \rho - k_2 \rho^2 - \frac{1}{|\sigma|^{1/2}} \times \\ & \left[\left(k_1 k_3 + \frac{k_1^3}{2} \right) |\sigma| - k_1^2 |\sigma|^{1/2} \operatorname{sgn}(\sigma) \rho + \left(k_1 k_4 + \frac{5k_1 k_2^2}{2} \right) \sigma^2 - 3k_1 k_2 \sigma \rho + \frac{k_1}{2} \rho^2 \right] \\ & + (-k_1 |\sigma|^{1/2} \operatorname{sgn}(\sigma) - k_2 \sigma + 2\rho) \frac{d}{dt} \varphi(t) \end{aligned}$$

Therefore, we can rewrite $\dot{V}_1(t)$ as

$$\dot{V}_1(t) = -\frac{1}{|\sigma|^{1/2}} \xi^T Q_1 \xi - \xi^T Q_2 \xi + w^T \frac{d}{dt} \varphi(t) \xi \quad (22)$$

where

$$Q_1 = \begin{pmatrix} k_1 k_3 + \frac{1}{2} k_1^3 & 0 & -\frac{1}{2} k_1^2 \\ 0 & k_1 k_4 + \frac{5}{2} k_1 k_2^2 & -\frac{3k_1 k_2}{2} \\ -\frac{1}{2} k_1^2 & -\frac{3k_1 k_2}{2} & \frac{1}{2} k_1 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} k_2 k_3 + 2k_1^2 k_2 & 0 & 0 \\ 0 & k_2 k_4 + k_2^3 & -k_2^2 \\ 0 & -k_2^2 & k_2 \end{pmatrix}$$

$w^T = [-k_1, -k_2, 2]^T$. Using the bounds condition on the perturbation (19), motivated by Ref. [21], then (22) can be rewritten as

$$\dot{V}_1(t) \leq -\frac{1}{|\sigma|^{1/2}} \xi^T \tilde{Q}_1 \xi - \xi^T \tilde{Q}_2 \xi$$

where

$$\tilde{Q}_1 = \begin{pmatrix} k_1 k_3 + \frac{1}{2} k_1^3 - k_1 \bar{\delta} & 0 & -\frac{1}{2} k_1^2 - \bar{\delta} \\ 0 & k_1 k_4 + \frac{5}{2} k_1 k_2^2 & -\frac{3k_1 k_2}{2} \\ -\frac{1}{2} k_1^2 - \bar{\delta} & -\frac{3k_1 k_2}{2} & \frac{1}{2} k_1 \end{pmatrix}, \quad \tilde{Q}_2 = \begin{pmatrix} k_2 k_3 + 2k_1^2 k_2 - k_2 \bar{\delta} & 0 & 0 \\ 0 & k_2 k_4 + k_2^3 & -k_2^2 \\ 0 & -k_2^2 & k_2 \end{pmatrix}$$

It is obvious that if $\tilde{Q}_1 > 0$, $\tilde{Q}_2 > 0$, then $\dot{V}_1(t)$ is negative definite. It is clear that if the parameters k_i , ($i = 1, 2, 3, 4$) satisfying the condition (15), then $\tilde{Q}_1 > 0$ and $\tilde{Q}_2 > 0$.

Noticing that $\lambda_{\min}(\tilde{Q}_i) \|\xi\|_2^2 \leq \xi^T \tilde{Q}_i \xi \leq \lambda_{\max}(\tilde{Q}_i) \|\xi\|_2^2$, ($i = 1, 2$), we have

$$\dot{V}_1(t) \leq -\frac{1}{|\sigma|^{1/2}} \lambda_{\min}(\tilde{Q}_1) \|\xi\|_2^2 - \lambda_{\min}(\tilde{Q}_2) \|\xi\|_2^2$$

Since

$$\|\xi\|_2^2 = |\sigma| + \sigma^2 + \rho$$

$$\lambda_{\min}(P) \|\xi\|_2^2 \leq V_1(t) \leq \lambda_{\max}(P) \|\xi\|_2^2$$

So, one can get

$$|\sigma|^{1/2} \leq \|\xi\|_2 \leq \frac{\sqrt{V_1(t)}}{\sqrt{\lambda_{\min}(P)}}$$

$$\frac{V_1(t)}{\lambda_{\max}(P)} \leq \|\xi\|_2^2 \leq \frac{V_1(t)}{\lambda_{\min}(P)}$$

Further, one obtains

$$\dot{V}_1(t) \leq -\frac{\sqrt{\lambda_{\min}(P)}}{\sqrt{V_1(t)}} \cdot \lambda_{\min}(\tilde{Q}_1) \cdot \frac{V_1(t)}{\lambda_{\max}(P)} - \lambda_{\min}(\tilde{Q}_2) \cdot \frac{V_1(t)}{\lambda_{\max}(P)}$$

$$= -\mu_1 V_1^{1/2}(t) - \mu_2 V_1(t) \leq -\mu_1 V_1^{1/2}(t)$$

According to Lemma 2, it follows easily that the Lyapunov function $V_1(t)$ and σ globally converge to zero in finite time $T_1 = 2V_1^{1/2}(0)/\mu_1$. Hence, the proof is completed.

When $\sigma \equiv 0$, for $t \geq T_1$, then we obtain the sliding mode dynamics (13), next, we will prove its asymptotic stability to origin.

Theorem 2 Consider the sliding mode dynamics (13), it is stable and the state trajectory will asymptotically converge to zero.

Proof Choosing the following Lyapunov function for system (13)

$$V_2(t) = |\psi| \tag{23}$$

Taking α order fractional derivative of (23), it yields

$$D^\alpha V_2(t) = D^\alpha |\psi|$$

If $\psi = 0$, then $D^\alpha |\psi| = 0$, if $\psi > 0$, according to Definition 3, one has

$$D^\alpha |\psi| = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{|\psi'|}{(t-\tau)^\alpha} d\tau = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\psi'}{(t-\tau)^\alpha} d\tau = D^\alpha \psi$$

Similarly, if $\psi < 0$, then

$$D^\alpha |\psi| = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{|\psi'|}{(t-\tau)^\alpha} d\tau = -\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\psi'}{(t-\tau)^\alpha} d\tau = -D^\alpha \psi$$

So, we have $D^\alpha |\psi| = \text{sgn}(\psi) D^\alpha \psi$. That is

$$D^\alpha V_2(t) = D^\alpha |\psi| = \text{sgn}(\psi) D^\alpha \psi \tag{24}$$

Substituting (13) into (24), we have

$$D^\alpha V_2(t) = \text{sgn}(\psi) \left[-\frac{\eta_2(\psi + \text{sgn}(\psi))}{\text{sgn}(\psi) + \eta_1} \right] = \frac{-\eta_2}{\text{sgn}(\psi) + \eta_1} (|\psi| + \text{sgn}^2(\psi)) \leq -\lambda |\psi| = -\lambda V_2(t) \tag{25}$$

where $\lambda = \frac{\eta_2}{1 + \eta_1}$. According to (25), we can assume that there exist a nonnegative function

$M(t)$ such that

$$D^\alpha V_2(t) + \lambda V_2(t) + M(t) = 0 \tag{26}$$

Taking the Laplace transform on (26), we get

$$s^\alpha V_2(s) - s^{\alpha-1} V_2(0) + \lambda V_2(s) + M(s) = 0$$

where $V_2(s) = \mathcal{L}\{V_2(t)\}$, $M(s) = \mathcal{L}\{M(t)\}$. Further, one has

$$V_2(s) = \frac{s^{\alpha-1} V_2(0) - M(s)}{s^\alpha + \lambda} \tag{27}$$

Taking the inverse Laplace transform on (27), it yields

$$V_2(t) = V_2(0)E_\alpha(-\lambda t^\alpha) - M(t) * [t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)]$$

Since both t^α and $E_{\alpha,\alpha}(-\lambda t^\alpha)$ are nonnegative [22], so

$$V_2(t) \leq V_2(0)E_\alpha(-\lambda t^\alpha)$$

According to Lemma 3, one obtains

$$|\psi| \leq |\psi(0)| \frac{K}{1 + |\lambda t^\alpha|}$$

where K is a positive constant. It is obvious that $\psi \rightarrow 0$ as $t \rightarrow \infty$. Thus, the proof is completed.

When $\psi \equiv 0$, we can obtain the reduced-order fractional order system (10), thus, the control problem of n-dimensional system is transformed to the stabilization problem of a reduced-order system.

3.2. Application of the Proposed Control Scheme for a Class of 3D Fractional-order Systems

To show the applicability of the proposed control approach for stabilizing other kinds of fractional-order systems, the following uncertain fractional-order systems are adopted here.

$$\begin{aligned} D^\alpha X &= f_1(X, y) \\ D^\alpha y &= f_2(X, y) + \Delta f(X, y) + d(t) + u(t) \end{aligned} \quad (28)$$

where $\alpha \in (0,1)$, $X \in R^2$, $y \in R$ are the state vectors, $f_1(X, y)$ and $f_2(X, y)$ are nonlinear functions, $\Delta f(X, y)$ and $d(t)$ are the model uncertainty and external disturbance, respectively, $u(t) \in R$ is the single control input to be designed later.

Assumption 2 The uncertainty term $\Delta f(X, y)$, external disturbance $d(t)$ are derivable, and the bounds of their derivatives is a known constant γ :

$$\left| \frac{d}{dt}(\Delta f(X, y) + d(t)) \right| \leq \gamma$$

Assumption 3 The function $f_1(X, y)$ is smooth in a neighborhood of $y = 0$, and the subsystem $D^\alpha X = f_1(X, 0)$ is asymptotically stable about the origin $X = 0$ for all X .

Remark 2 System (28) is very common, most of canonical fractional-order system satisfy this form, such as fractional-order Lu system, fractional-order Genesio-Tesi system, fractional-order Chen system etc. It is apparent that the three-dimensional uncertain fractional-order chain of integrator (as given by system (6)) is the special case of system (28).

We can establish the following sliding mode surface for system (28)

$$\psi = y + cI^\alpha y$$

where $c > 0$, I^α is the fractional integral operator of order α , defined in (1).

To obtain the desired dynamics, another sliding mode variable σ is designed, which is same as (11). Through similar design procedures, we can get the robust control law $u(t)$, which has the following form:

$$\begin{aligned} u(t) &= -[f_2(X, y) + cy + (\text{sgn}(\psi) + \eta_1)^{-1}(\eta_2(y + cI^\alpha y + \text{sgn}(\psi))) \\ &\quad + k_1|\sigma|^{1/2} \text{sgn}(\sigma) + k_2\sigma - z] \\ \dot{z} &= -k_3 \text{sgn}(\sigma) - k_4\sigma \end{aligned} \quad (29)$$

Correspondingly, we have the SOSM controller:

$$\begin{aligned}\dot{\sigma} &= -k_1 |\sigma|^{1/2} \operatorname{sgn}(\sigma) - k_2 \sigma + \rho \\ \dot{\rho} &= -k_3 \operatorname{sgn}(\sigma) - k_4 \sigma + \frac{d}{dt} \varepsilon(t)\end{aligned}\tag{30}$$

where $\left| \frac{d}{dt} \varepsilon(t) \right| \leq (1 + \eta_1) \left| \frac{d}{dt} (\Delta f(X, y) + d(t)) \right| = \bar{\gamma}$. Through the same operations presented in subsection 3.1, we known that σ and ψ will converge to zero. Similarly, when $\psi \equiv 0$, we obtain

$$\dot{y} = -c I^\alpha y\tag{31}$$

Taking α order fractional derivative of (31), it yields

$$D^\alpha y = -cy\tag{32}$$

Since $c > 0$, according to Lemma 1, we know that y will converge to origin asymptotically, then according to Assumption 3, we conclude that the subsystem $D^\alpha X = f_1(X, y)$ is asymptotically stable too. So, under the control of (29), the desired sliding mode dynamics (32) can be achieved. In other word, the proposed control scheme converts the control problem of three-dimensional system to the equivalent stabilization problem of single state variable y .

4. Simulation Results

In this section, two examples are given to verify the effectiveness and feasibility of the proposed SMC scheme.

4.1. Example 1

Consider the following uncertain fractional-order Arneodo system

$$\begin{aligned}D^\alpha x_1 &= x_2 \\ D^\alpha x_2 &= x_3 \\ D^\alpha x_3 &= 5.5x_1 - 3.5x_2 - 0.4x_3 - x_1^3 + \Delta f(x) + d(t) + u(t)\end{aligned}\tag{33}$$

the uncertainty terms of the system are selected as follows:

$$\Delta f(x) + d(t) = 0.25 \cos(2t) - 0.1 \sin(t)$$

In this simulation, when the initial conditions are selected as $x(0) = [3, 1, 1]^T$, and the fractional order α is set as 0.98, system (33) can behave chaotically, the chaotic attractors are shown in Figure 1.

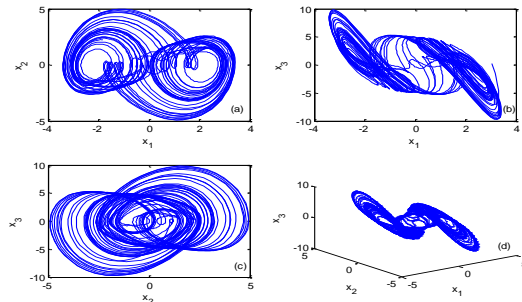


Figure 1. Chaotic Attractors of Fractional-order Arneodo System (33)

To observe the control effect of the designed controller (14), the state trajectories of (33) without control are given, displayed in Figure 2.

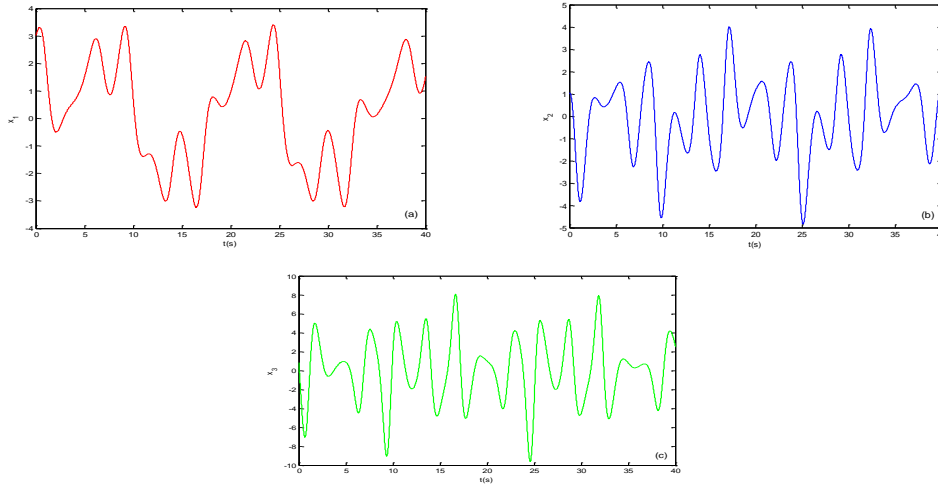


Figure 2. Time Responses of the System (33) without Control

Selecting the control parameters as $c_1 = c_2 = 5$, $\eta_1 = 1.5$, $\eta_2 = 8$, $k_1 = 1$, $k_2 = 3$, $k_3 = 2$, $k_4 = 35$, then we can obtain two appropriate sliding surfaces ψ , σ and a robust chattering-free SOSM controller. When the controller is activated at $t = 10$ s, we can gain the desired performance of system (33), illustrated in Figure 3.

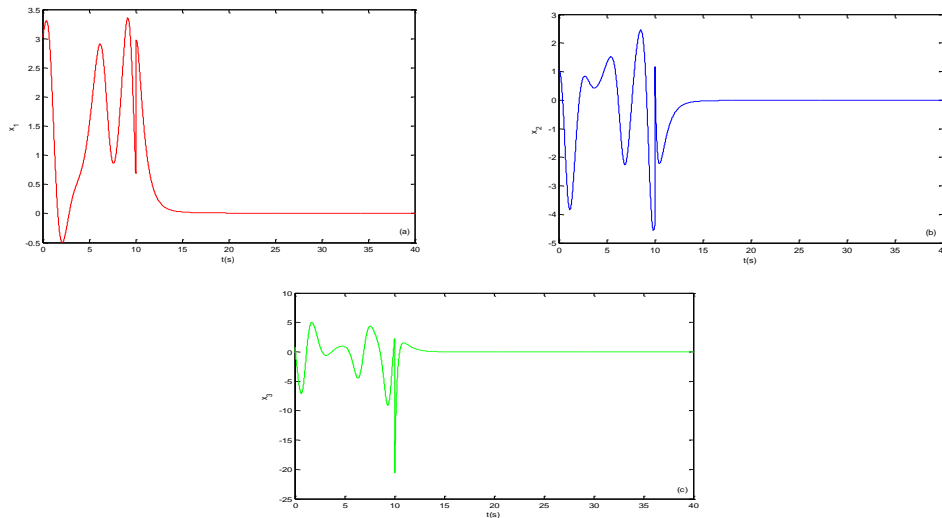


Figure 3. Time Responses of the System (33) with Controller Activated at $t = 10$ s

One can see from Figure 3 that the state trajectories converge to zero asymptotically, which implies that applying the proposed control scheme, system (33) can be stabilized. The time response of the sliding surface ψ is depicted in Figure 4.

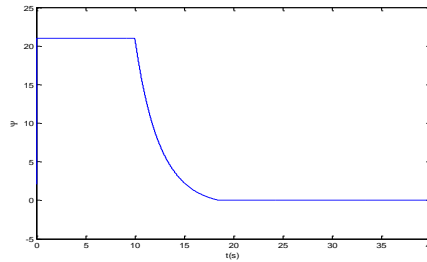


Figure 4. Time Response of ψ with the Controller Activated at $t = 10$ s

In Figure 4, ψ tend to zero asymptotically. These simulation results sufficiently verified the effectiveness of the proposed method.

4.2. Example 2

In this example, the control law (29) is used to stabilize the uncertain fractional-order Lorenz system, which satisfies the form (28), described by

$$\begin{aligned} D^\alpha x_1 &= 10(x_2 - x_1) \\ D^\alpha x_2 &= 28x_1 - x_2 - x_1x_3 \\ D^\alpha x_3 &= x_1x_2 - \frac{8}{3}x_3 \end{aligned} \quad (34)$$

when $\alpha = 0.995$, the initial values are selected as $x(0) = [-1, 2, 3]^T$, system (34) behave chaotically, the attractors and state trajectories are shown in Figure 5 and 6, respectively.

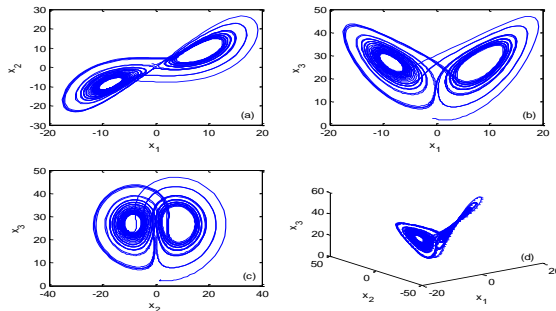


Figure 5. Chaotic Attractors of Fractional-order Lorenz System (34)

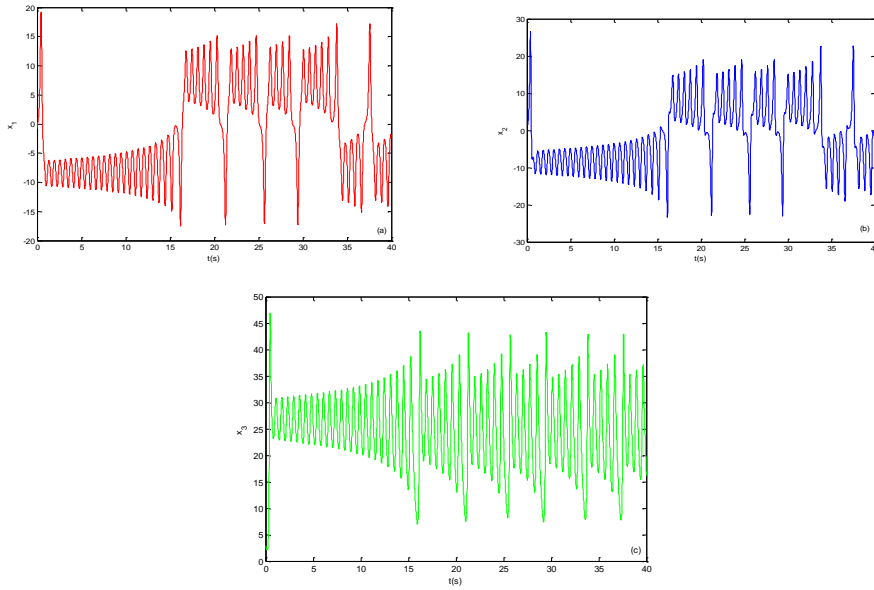


Figure 6. Time Responses of the System (34)

It is clear that if $x_2 = 0$, the following subsystem of system (34)

$$\begin{aligned} D^\alpha x_1 &= 10(x_2 - x_1) \\ D^\alpha x_3 &= x_1 x_2 - \frac{8}{3} x_3 \end{aligned} \quad (35)$$

is asymptotically stable at the origin $x_1 = 0$ and $x_3 = 0$ for all x_1 and x_3 . Hence, we can assume $X = [x_1, x_3]^T$ and $y = x_2$, system (35) satisfies Assumption 3. Consequently, the controlled uncertain system (34) can be written as follows

$$\begin{aligned} D^\alpha x_1 &= 10(x_2 - x_1) \\ D^\alpha x_2 &= 28x_1 - x_2 - x_1 x_3 + \Delta f(x) + d(t) + u(t) \\ D^\alpha x_3 &= x_1 x_2 - \frac{8}{3} x_3 \end{aligned} \quad (36)$$

where $\Delta f(x) + d(t) = 0.2 \sin(3t) - 0.15 \cos(2t)$, we choose the control parameters as $c = 2$, $\eta_1 = 1.5$, $\eta_2 = 1.5$, $k_1 = 10$, $k_2 = 3$, $k_3 = 30$, $k_4 = 35$, then we can obtain two appropriate sliding surfaces ψ , σ and a robust chattering-free SOSM controller. When the controller is activated at $t = 10$ s, the desired performance of system (36) is illustrated in Figure 7.

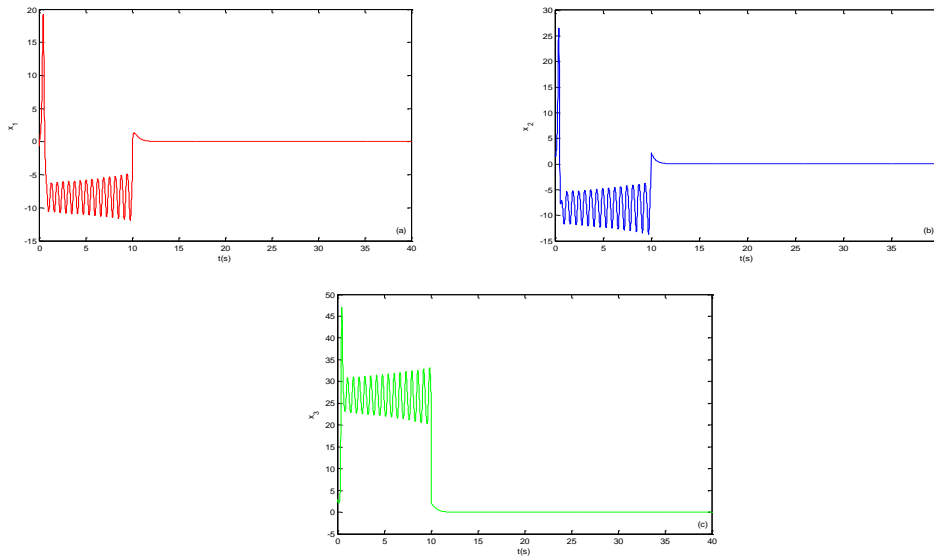


Figure 7. Time Responses of the System (36) with the Controller Activated at $t = 10$ s

It is clear that the chaos is suppressed and the state variables converge to zero asymptotically, which implies that the proposed control law (29) is feasible. The time evolution of the sliding surface ψ is shown in Figure 8. These simulation experiments demonstrated our theoretical results.

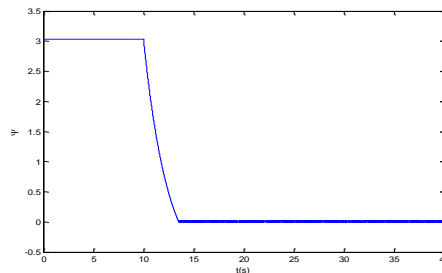


Figure 8. Time Evolution of ψ with the Controller Activated at $t = 10$ s

5. Conclusions

A novel SMC scheme is proposed to stabilize a class of uncertain fractional-order chaotic systems. In this paper, first, a simple sliding surface is designed, which transformed the control task of n-dimensional system to the equivalent stabilization problem of a reduced-order system, the control process became easier. Subsequently, on the basis of SOSM technique and finite-time control theory, a robust chattering-free controller is designed to ensure the existence of the sliding motion in a given time. The stability of the suggested scheme has been demonstrated via traditional and fractional Lyapunov stability theory, which proven that both reaching phases of two sliding surfaces are stable. Two simulation examples are provided to demonstrate the effectiveness of the proposed method.

Acknowledgements

The author would like to thank the reviewers for their careful reading and providing some pertinent suggestions. This work is supported by the National Natural Science Foundation of China (61273119, 61374038) and the Nature Science Foundation of Jiangsu Province of China (BK2011253).

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