

## Multi-soliton Solutions for a Class of Fifth-order Evolution Equations

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### Abstract

By employing a simplified bilinear method, a class of generalized fifth-order KdV (gfKdV) equations which arise in nonlinear lattice, plasma physics and ocean dynamics are investigated. With the aid of symbolic computation, both solitary wave solutions and multiple-soliton solutions are obtained. These new exact solutions will extend previous results and help us explain the properties of nonlinear solitary waves in many physical models in shallow water. Parametric analysis is carried out in order to illustrate that the soliton amplitude, width and velocity are affected by the coefficient parameters in the equation.

**Keywords:** Multiple soliton solutions; Fifth-order evolution equations; Cole-Hopf transformation; Hirota bilinear method

### 1. Introduction

In this paper, the simplified version of Hirota's method is applied to construct multi-soliton solutions of a class of fifth-order evolution equations of the form

$$u_t + \alpha uu_{3x} + \beta u_x u_{2x} + \gamma u^2 u_x + u_{5x} \quad (1)$$

where the coefficients  $\alpha, \beta, \gamma$  are real-valued parameters that are physically meaningful.

This class includes the Lax [9], Sawada-Kotera (SK) [10, 13, 14], Kaup-Kupershmidt (KK) [11] and Ito equations [7]. As the constants  $\alpha, \beta, \gamma$  take different values, the properties of Eq. (1) drastically change. For instance, the Lax equation with  $\alpha = 10, \beta = 20$  and  $\gamma = 30$ , and the SK equation where  $\gamma = \beta = \alpha = 5$ , are completely integrable. These two equations have N-soliton solutions and an infinite set of conservation laws. The KK equation, with  $\alpha = 10, \beta = 25 = 10$  and  $\gamma = 20$ , is also known to be integrable [8] and to have bilinear representations. A fourth equation in the class (1) is the Ito equation with  $\alpha = 3, \beta = 6$  and  $\gamma = 2$ , which is not completely integrable, but has a limited number of conservation laws [7].

Obviously, for arbitrary values of  $\alpha, \beta$  and  $\gamma$  Eq. (1) is not completely integrable, and therefore does not admit solitons. This does, however, not exclude the existence of closed-form solitary-wave solutions. Also, with scales on  $u, x$  and  $t$ , the named equations cannot be transformed into one another; they are fundamentally different.

To our knowledge, there have been no discussions on Eq. (1) in the present of the coefficient parameters  $\alpha, \beta, \gamma$ . All the existing works deal with special cases of Eq. (1). Considering such insufficiency, we will apply a simplified bilinear method to investigate the N-soliton solutions for Eq. (1). The results obtained for the multiple regular soliton solutions

agree well with the results in the previous works and this the first work for the multiple singular soliton solutions.

In this work, the Cole-Hopf transformation method combined with Hirota's bilinear method is used to study the equations in class (1). Our goal from applying this method is to construct multiple regular soliton solutions and multiple singular soliton solutions. The introduced simplified algorithm derives the auxiliary functions, obtained in Hirota's method, without using the bilinear forms.

## 2. Soliton Solutions for Class (1)

### 2.1. Cole-Hopf Transformation Method

To derive N-soliton solutions of any completely integrable equation, we will mainly use the Hirota's direct method. The Hirota method [4, 12, 5, 11] relies on a transformation for considered equation to a bilinear form. The bilinear forms are usually used to enable us deriving the auxiliary function. It is remarkable to mention that it is not easy for us to find the bilinear form for many equations and sometimes it requires the introduction of new dependent and sometimes even independent variables. However, Hereman *et al.*, [15, 16, 3], formally introduced the simplified algorithm to derive the auxiliary functions without using the bilinear forms. The Cole-Hopf transformation method combined with the simplified Hirota's sense is a powerful method to determine multiple soliton solutions and multiple singular soliton solutions for integrable systems. We summarize the necessary steps of the method [1, 2, 6, 19, 18, 17]:

We first substitute

$$u(x, t) = e^{k_i x - w_i t}$$

into the linear terms of the equation under discussion to determine the dispersion relation between  $k_i, r_i$  and  $w_i$ . To obtain the single soliton solution, we use a suitable transformation method, such as the Cole-Hopf transformation

$$u(x, t) = R(\ln f)_{xx}$$

into the equation under discussion, where the auxiliary function  $f(x, t)$  is given by

$$f(x, t) = 1 + e^{\theta_1}$$

Where

$$\theta_i = k_i x - w_i t, \quad i = 1, 2, \dots, N$$

Then solving the resulting equation to determine the numerical value for R. The N-soliton solutions can be obtained by using the following forms for  $f(x, t)$ :

For dispersion relation, we use

$$u(x, t) = e^{k_i x - w_i t}$$

For single soliton solution, we use

$$f(x, t) = 1 + e^{\theta_1}$$

- For two-soliton solutions, we use

$$f(x, t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}$$

- For three-soliton solutions, we use

$$f(x, t) = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3} + a_{123}e^{\theta_1+\theta_2+\theta_3}$$

Notice that if we find that  $a_{123} = a_{12}a_{13}a_{23}$  then three-soliton solutions are obtained. The determination of three-soliton solutions confirms the fact that N-soliton solutions exist for any order.

However, for the multiple singular soliton solutions, we follow the following steps:

- For dispersion relation, we use

$$u(x, t) = e^{k_i x - w_i t}$$

- For single soliton, we use

$$f(x, t) = 1 - e^{\theta_1}$$

- For two-solution solutions, we use

$$f(x, t) = 1 - e^{\theta_1} - e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}$$

- For three-soliton solutions, we use

$$f(x, t) = 1 - e^{\theta_1} - e^{\theta_2} - e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3} - a_{123}e^{\theta_1+\theta_2+\theta_3}$$

## 2.2. Multiple Soliton Solutions

We first substitute

$$u(x, t) = e^{k_i x - w_i t} \tag{2}$$

into the linear terms of Eq.(1) to determine the dispersion relation as follows

$$w_i = k_i^5 \tag{3}$$

It is obvious that the dispersion relation (3) depends on the coefficient of x. As a result we obtain

$$\theta_i = k_i x - k_i^5 t. \tag{4}$$

The single soliton solution of Eq. (1) is assumed to be

$$u(x, t) = R(\ln f)_{xx} \tag{5}$$

where the auxiliary function  $f(x; t)$  is given by

$$f(x, t) = 1 + e^{\theta_1} = 1 + e^{k_1 x - k_1^5 t} \quad (6)$$

Substitute (5) into Eq. (1) and solve the system for R. We find the solvability condition

$$\alpha^2 + \beta\alpha - 10\gamma = 0 \quad (7)$$

is necessary to obtain

$$R = \frac{60}{\alpha + \beta} \quad (8)$$

By substituting  $f(x, t)$  into (5), we obtain the single soliton solution

$$\begin{aligned} u(x, t) &= \frac{60}{\alpha + \beta} k_1^2 \frac{e^{\theta_1}}{(1 + e^{\theta_1})^2} \\ &= \frac{60}{\alpha + \beta} k_1^2 \frac{e^{k_1 x - k_1^5 t}}{(1 + e^{k_1 x - k_1^5 t})^2} \end{aligned} \quad (9)$$

The solution in (9) can be written as

$$u(x, t) = \frac{15}{\alpha + \beta} k_1^2 \operatorname{sech}^2 \left( \frac{\theta_1}{2} \right) \quad (10)$$

which is a bell-shaped solitary wave solution for Eq. (1). For two-soliton solutions, we set

$$f(x, t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2} \quad (11)$$

where  $\theta_1$  and  $\theta_2$  are defined in (4). Using (11) in

$$u(x, t) = \frac{60}{\alpha + \beta} (\ln f)_{xx} \quad (12)$$

and substituting the result in Eq.(1), we obtain the phase shift by

$$a_{12} = K(\alpha, \beta, k_1, k_2) \quad (13)$$

The three-soliton solutions can be obtained by using

$$\begin{aligned} f(x, t) &= 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12} e^{\theta_1 + \theta_2} + \\ &a_{13} e^{\theta_1 + \theta_3} + a_{23} e^{\theta_2 + \theta_3} + a_{123} e^{\theta_1 + \theta_2 + \theta_3} \end{aligned} \quad (14)$$

where  $a_{ij}$  is given by (13). Substituting (14) and (12) into Eq.(1), we find always that

$$a_{123} = a_{12} a_{13} a_{23} \quad (15)$$

Since (15) holds, we can use the results indicated in (6,7,8,9) to conclude that the N-soliton solutions for Eq.(1) can be obtained for finite N, where  $N \geq 1$ .

### 2.3. Multiple Singular Soliton Solutions

To obtain a single singular soliton solution, we substitute

$$u(x, t) = e^{k_i x - w_i t} \quad (16)$$

into the linear terms of Eq.(1). This gives the dispersion relation by

$$w_i = k_i^5 \tag{17}$$

and as a result we obtain

$$\theta_i = k_i x - k_i^5 t. \tag{18}$$

The singular single soliton solution of Eq (1) is assumed to be

$$u(x, t) = R(\ln f)_{xx} \tag{19}$$

where the auxiliary function  $f(x, t)$  is given by

$$f(x, t) = 1 + e^{\theta_1} = 1 + e^{k_i x - k_i^5 t} \tag{20}$$

Substituting (19)-(20) into Eq.(1) and solving for R we find  $R = 60/(\alpha + \beta)$ . Now, Eq.(19) gives the single singular soliton solution

$$u(x, t) = -\frac{60}{\alpha + \beta} k_1^2 \frac{e^{\theta_1}}{e^{\theta_1 - 1}},$$

and so

$$u(x, t) = \frac{15}{\alpha + \beta} k_1^2 \operatorname{csc} h^2 \left( \frac{\theta_1}{2} \right)$$

where  $\theta_1$  is defined by (18).

Multiple singular soliton solutions for Eq. (1) can be expressed in the following form:

$$u(x, t) = \frac{60}{\alpha + \beta} (\ln f)_{xx}. \tag{21}$$

To determine the two singular soliton solutions explicitly, we substitute

$$f(x, t) = 1 - e^{\theta_1} - e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2} \tag{22}$$

where  $\theta_1$  and  $\theta_2$  are defined in (18), into (21). As a result we get

$$a_{ij} = K(\alpha, \beta, k_1, k_2).$$

For the singular three-soliton solutions we use

$$f(x, t) = 1 - e^{\theta_1} - e^{\theta_2} - e^{\theta_3} + a_{12} e^{\theta_1 + \theta_2} + a_{13} e^{\theta_1 + \theta_3} + a_{23} e^{\theta_2 + \theta_3} - a_{123} e^{\theta_1 + \theta_2 + \theta_3}, \tag{23}$$

where  $\theta_i$  ( $i = 1, 2, 3$ ) are defined in (18) and  $a_{ij}$  are defined in (15). Proceeding as before, we obtain always

$$a_{123} = a_{12} a_{13} a_{23}$$

The singular three-soliton solution can be obtained explicitly using (21) for the function  $f$  in (23).

### 2.3. Analysis of the Parameters

The solution (10) gives a profile of bell-shaped solitary wave with soliton amplitude  $\Delta$  and width  $\Delta$  can be expressed as

$$\text{amp} = \frac{15}{|\alpha + \beta|} k_1^2, \quad \Delta = \frac{2}{|k_1|} \quad (24)$$

With the characteristic-line method [2, 6], the characteristic line for each solitary wave can be defined by

$$x - k_i^4 = 0, \quad i = 1, 2, 3, \dots, \quad (25)$$

which can be derived from relations (3) and (4). Correspondingly, the velocity  $v$  of each solitary wave can be expressed by

$$v_i = k_i^4 \quad (26)$$

Further, the absolute value of velocity  $v$  determines the speed, namely, velocity in magnitude, and propagation direction of soliton is decided by the sign of  $v$ .

We now analytically examine the effects of the parameters  $\alpha, \beta, \gamma$  on the behavior of the solitary waves. The soliton amplitude  $\text{amp}$  is dependent on the ratio of 15 to  $|\alpha + \beta|$ . The solitonic amplitude increases with an increase in the ratio  $15/|\alpha + \beta|$ . However, Expression (26) indicates that the propagation velocity of the solitary wave is not influenced by the coefficient parameters  $\alpha, \beta$  and  $\gamma$ .

### 3. Applications

In this section we will present some examples and test cases.

#### 3.1. The Sawada-Kotera Equation

Consider the Sawada-Kotera equation

$$u_t + 15uu_{3x} + 15u_xu_{2x} + 45u^2u_x + u_{5x} = 0 \quad (27)$$

We can verify that the parameters  $\alpha, \beta$  and  $\gamma$  satisfy the solvability Condition (7). So,  $R = 2$  and the one-soliton solution is given by

$$u(x, t) = \frac{1}{2} k_1^2 \operatorname{sech}^2 \left( \frac{\theta_1}{2} \right)$$

and the singular one-soliton solution is

$$u(x, t) = \frac{1}{2} k_1^2 \operatorname{csc}^2 \left( \frac{\theta_1}{2} \right),$$

where  $\theta_1$  is defined in (4).

Using (11) in

$$u(x, t) = 2(\ln f)_{xx} \quad (28)$$

and substituting the result in Eq. (27), we obtain the phase shift by

$$a_{12} = \frac{(k_1^2 - k_1k_2 + k_2^2)(k_1 - k_2)^2}{(k_1^2 + k_1k_2 + k_2^2)(k_1 + k_2)^2} \quad (29)$$

We can generalize for other phase shifts by

$$a_{ij} = \frac{(k_i^2 - k_i k_j + k_j^2)(k_i - k_j)^2}{(k_i^2 + k_i k_j + k_j^2)(k_i + k_j)^2}, \quad 1 \leq i \leq j \leq 3. \quad (30)$$

Using (11) and (14) in (28), we can obtain the closed-form two-soliton and three-soliton solutions.

### 3.2. The Lax Equation

The Lax equation is given by

$$+ 10uu_{3x} + 20u_x u_{2x} + 30u^2 u_x + u_{5x} = 0 \quad (31)$$

The parameters  $\alpha, \beta$  and  $\gamma$  satisfy the solvability Condition (7), and as a result,  $R = 2$ . For the Lax equation, the one-soliton solution is given by

$$u(x, t) = \frac{1}{2} k_1^2 \operatorname{sech}^2 \left( \frac{\theta_1}{2} \right)$$

and the singular one-soliton solution is

$$u(x, t) = \frac{1}{2} k_1^2 \operatorname{csch}^2 \left( \frac{\theta_1}{2} \right),$$

where  $\theta_1$  is defined in (4).

For the phase shift we find that

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$$

and for other phase shifts we write

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}$$

As a result, we can easily obtain explicit forms for the N-soliton solutions for the Lax Eq.(31)

### 4. Conclusion

In this paper, with the aid of symbolic computation, we have investigated a class of fifth-order evolution equations. By the Cole-Hopf transformation method combined with Hirota's bilinear method, N-soliton solution and singular N-soliton solution for Eq. (1) have been obtained. Furthermore, based on the one-soliton solution in expression (10), we have carried out the parametric analysis in order to investigate the effects of the parameters  $\alpha, \beta, \gamma$  on the soliton amplitude, width and velocity.

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