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Bayesian Estimation and Prediction of Burr Type XI Distribution under Singly and Doubly Censored Samples

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Abstract

The purpose of the paper is to address the problem of estimation and prediction of the Burr type XI distribution under Bayesian framework based on censored samples. Five informative and non-informative priors have been assumed under five different (symmetric and asymmetric) loss functions for posterior analysis. The expressions for Bayes estimators, posterior risks, credible intervals, posterior predictive intervals have been derived and evaluated. The simulation study has been carried out in order to assess and compare the performance of Bayesian point and interval estimators. The study indicated that for Bayesian estimation and prediction of the said distribution, the gamma prior along with quadratic loss function can efficiently be employed.

Keywords: *Bayes estimators, posterior risks, loss functions, censoring*

1. Introduction

Burr (1942) introduced twelve forms of the Burr distribution. However, most of the authors have dealt with Bayesian and classical estimation of Burr type X and XII distributions. Unfortunately, the Burr type XI distribution hasn't been considered by any author. Surles and Padgett (2001) are among the most recent authors who have introduced two-parameter Burr Type X distribution, which can also be described as generalized Rayleigh distribution. Shao (2004a) discussed maximum likelihood estimation for the three-parameter Burr type XII distribution. Shao *et al.* (2004b) studied models for extremes using the extended three parameter Burr type XII distribution with application to flood frequency analysis. According to Soliman (2005), this distribution covers the curve shape characteristics for a large number of distributions. The versatility and flexibility of the Burr type XII distribution turns it quite attractive as a tentative model for data whose underlying distribution is unknown. Wahed (2006) presented Bayes estimators for the parameters of Burr type XII distribution under the symmetric squared error loss function and the asymmetric linear exponential loss function based on a simple prior distribution. Wu *et al.* (2007) considered the estimation problems for Burr distribution based on progressive type II censoring with random removals, where the number of units removed at each failure time has a discrete uniform distribution. Silva *et al.* (2008) proposed a location-scale regression model based on Burr XII distribution and referred it as the log-Burr XII regression model. Dasgupta (2011) discussed that under certain conditions, the distribution of Burr can be shown to follow an extreme value distribution.

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Makhdoom and Jafari (2011) obtained Bayesian estimators for the shape parameter of the Burr Type XII distribution using grouped and un-grouped data. Panahi and Asadi (2011) considered the statistical inferences based on a Type-II hybrid censored sample from a Burr type XII distribution. The authors considering analysis of Burr type X and XII distributions include: Mousa and Jaheen (2002), Soliman (2002), Soliman (2005), Wu and Yu (2005), Amjad and Ayman (2006), Aludaat *et al.* (2008) and Yarmohammadi and Pazira (2010).

In practice, the population from which the sample is drawn can be incomplete, that is, all information regarding a portion of the population is omitted or do not exist. That means the observations have been censored and this process is called censoring. Censoring is very important technique which is mostly used in life testing experiments. Fauzy (2004) discussed the classical interval estimation for parameters of exponential distribution under doubly type II censoring. Akhter and Hirai (2009) estimated the scale parameter from the Rayleigh distribution from type II singly and doubly censored data. Yarmohammadi and Pazira (2010) obtained Bayesian and classical estimators for the shape parameter, reliability and failure rate functions of the Generalized-Exponential distribution using complete and type-II censored samples. AL-Hussaini and Hussein (2011) presented the maximum likelihood and Bayes estimators of the parameters, survival function (SF) and hazard rate function (HRF) for the three-parameter exponentiated Burr type XII distribution under type II censored scheme. Feroze and Aslam (2012) addressed the problem of Bayesian analysis of the parameter of Burr type X distribution under complete and censored samples. Feroze and Aslam (2012) considered the Bayesian estimation of the Gumbel type II distribution based and doubly type II censored samples.

The rare analysis of the Burr type XI distribution under any estimation technique motivated the authors to discuss the Bayesian estimation and prediction of the said distribution based on singly and doubly type II censored samples.

The probability density function (pdf) of Burr type XI distribution is:

$$f\left(x,\theta\right) \propto \theta\left\{x - \left(\frac{1}{2\pi}\right)Sin\left(2\pi x\right)\right\}^{\theta} \quad ;\ x>0\,, \qquad \theta>0 \tag{1}$$

The cumulative distribution function (CDF) of the distribution is:

$$F\left(x,\theta\right) = \left\{x - \left(\frac{1}{2\pi}\right)Sin\left(2\pi x\right)\right\}^{\theta} \tag{2}$$

2. Materials and Methods

In this section, the posterior distributions, Bayes estimators, posterior risks, credible intervals, posterior predictive distributions and posterior predictive intervals have been derived under five priors and loss functions based on singly and doubly type II censored samples. These point and interval estimators have been evaluated in the coming sections.

Prior Distributions

The main difference between the concepts of Bayesian and classical approaches it the use of prior information under the Bayesian framework. The prior distribution combines with the likelihood function to produce the posterior distribution. In this way the Bayesian approach updates the current information with inclusion of prior information. However, the choice of suitable prior has always been a great task under Bayesian framework. Sometimes a prior distribution can be approximated by one that is in a convenient family of distributions, which combines with the likelihood to produce a posterior that is manageable. But in real life such priors may not exist. In such situations, the researchers have to go for a non-informative prior. We have considered both informative and non-informative priors for the Bayesian analysis of the parameter of Burr type XI distribution.

One of the most widely used non-informative priors is a uniform prior.

It is defined as: $p(\theta) \propto 1\ ,\ \theta > 0$ (3)

Another non-informative prior which is frequently used in situations where one does not have much information about the parameters. This is defined as the distribution of the parameters proportional to the square root of the determinants of the Fisher information matrix, *i.e.*,

$p_j \propto \sqrt{|I(\theta)|}$ Where $|I(\theta)|$ is Fisher information matrix.

Here the Jeffreys prior for the parameter of Burr type XI distribution is:

$$p_j \propto \sqrt{|I(\theta)|} = \frac{1}{\theta}\ ,\ \theta > 0 \tag{4}$$

If prior information exists about parameter θ, then it should be utilized in the prior distribution of θ. For example, if the present model form is similar to a prior model form, and the present model is proposed to be a rationalized version based on more existing data, then the posterior distribution of θ from the prior model may be utilized as the prior distribution of θ for the current model. We have assumed gamma, chi square and exponential priors as informative priors for the Bayesian analysis of the parameter of the Burr type XI distribution. The description of the said priors is as under:

The exponential prior is assumed to be:

$$p(\theta) \propto \exp(-k\theta) \qquad ;\ \theta > 0, k > 0 \tag{5}$$

Where k is a hyper-parameter.

The gamma prior is assumed to be:

$$p(\theta) \propto \theta^{a-1} e^{-\theta b} \quad ;\theta > 0,\ a,b\ >\ 0 \tag{6}$$

Where; a and b are hyper-parameters.

The chi square prior is assumed to be:

$$p(\theta) \propto \theta^{\frac{h}{2}-1} e^{-\frac{\theta}{2}} \qquad ;\theta > 0;\ h \in N \tag{7}$$

Where; h is the hyper-parameter.

Bayesian Analysis for Singly Type II Censored Samples

Suppose ‘n’ items are put on a life-testing experiment and only first ‘r’ failure times have been observed, that is, $x_1 < x_2 < ... < x_r$ and remaining ‘n – r’ items are still working. Under the assumptions that the lifetimes of the items are independently and identically distributed Burr type XI random variable, the likelihood function of the observed data without the multiplicative constant can be written as:

$$L\left(\theta|\underline{x}\right) \propto \left[\prod_{i=1}^{r} f\left(x_i\right)\right]\left[1-F\left(x_r\right)\right]^{n-r} \tag{8}$$

$$L\left(\theta|\underline{x}\right) \propto \sum_{j=o}^{n-r}\left(-1\right)^j \binom{n-r}{j}\theta^r e^{-\theta\psi(x)} \tag{9}$$

Where $\psi\left(x\right) = \sum_{i=1}^{r}\ln\left\{x_i - \left(\frac{1}{2\pi}\right)Sin\left(2\pi x_i\right)\right\}^{-1} + j\ln\left\{x_r - \left(\frac{1}{2\pi}\right)Sin\left(2\pi x_r\right)\right\}^{-1}$

The posterior distribution for singly type II censored samples under the assumption of uniform, Jeffreys, exponential, gamma and chi square priors are respectively derived as:

$$p\left(\theta|\underline{x}\right) = \frac{\sum_{j=0}^{n-r}\left(-1\right)^j\binom{n-r}{j}\theta^r e^{-\theta\psi(x)}}{\Gamma\left(r+1\right)\sum_{j=0}^{n-r}\left(-1\right)^j\binom{n-r}{j}\left\{\psi\left(x\right)\right\}^{-(r+1)}}, \theta > 0 \tag{10}$$

$$p\left(\theta|\underline{x}\right) = \frac{\sum_{j=0}^{n-r}\left(-1\right)^j\binom{n-r}{j}\theta^{r-1} e^{-\theta\psi(x)}}{\Gamma\left(r\right)\sum_{j=0}^{n-r}\left(-1\right)^j\binom{n-r}{j}\left\{\psi\left(x\right)\right\}^{-(r)}}, \theta > 0 \tag{11}$$

$$p\left(\theta|\underline{x}\right) = \frac{\sum_{j=0}^{n-r}\left(-1\right)^j\binom{n-r}{j}\theta^r e^{-\theta\left\{\psi(x)+k\right\}}}{\Gamma\left(r+1\right)\sum_{j=0}^{n-r}\left(-1\right)^j\binom{n-r}{j}\left\{\psi\left(x\right)+k\right\}^{-(r+1)}}, \theta > 0 \tag{12}$$

$$p\left(\theta|\underline{x}\right) = \frac{\sum_{j=0}^{n-r}\left(-1\right)^j\binom{n-r}{j}\theta^{r+a-1} e^{-\theta\left\{\psi(x)+b\right\}}}{\Gamma\left(r+a\right)\sum_{j=0}^{n-r}\left(-1\right)^j\binom{n-r}{j}\left\{\psi\left(x\right)+b\right\}^{-(r+a)}}, \theta > 0 \tag{13}$$

$$p\left(\theta|\underline{x}\right)=\frac{\sum_{j=0}^{n-r}(-1)^{j}\binom{n-r}{j}\theta^{r+\frac{h}{2}-1}e^{-\theta\left\{\psi(x)+\frac{1}{2}\right\}}}{\Gamma\left(r+\frac{h}{2}\right)\sum_{j=0}^{n-r}(-1)^{j}\binom{n-r}{j}\left\{\psi(x)+\frac{1}{2}\right\}^{-\left(r+\frac{h}{2}\right)}},\theta>0 \tag{14}$$

The expressions for Bayes estimators and associated posterior risks have been derived on the basis of squared error loss function (SELF), quadratic loss function (QLF), weighted loss function (WLF), LINEX loss function (LLF) and Precautionary loss function (PLF) and are presented in the following. The expressions for each loss function have been discriminated by attaching the abbreviations of the corresponding loss function to subscripts of θ.

The Bayes estimators for singly type II censored samples based on uniform prior using SELF, QLF, WLF, LLF and PLF are:

$$\theta_{SELF}=\frac{(r+1)\sum_{j=0}^{n-r}(-1)^{j}\binom{n-r}{j}\{\psi(x)\}^{-(r+2)}}{\sum_{j=0}^{n-r}(-1)^{j}\binom{n-r}{j}\{\psi(x)\}^{-(r+1)}}\quad \theta_{QLF}=\frac{(r-1)\sum_{j=0}^{n-r}(-1)^{j}\binom{n-r}{j}\{\psi(x)\}^{-r}}{\sum_{j=0}^{n-r}(-1)^{j}\binom{n-r}{j}\{\psi(x)\}^{-(r-1)}}$$

$$\theta_{WLF}=\left\{\frac{\sum_{j=0}^{n-r}(-1)^{j}\binom{n-r}{j}\{\psi(x)\}^{-r}}{r\sum_{j=0}^{n-r}(-1)^{j}\binom{n-r}{j}\{\psi(x)\}^{-(r+1)}}\right\}^{-1}\quad \theta_{LLF}=-\ln\left\{\frac{\sum_{j=0}^{n-r}(-1)^{j}\binom{n-r}{j}\{\psi(x)+1\}^{-(r+1)}}{\sum_{j=0}^{n-r}(-1)^{j}\binom{n-r}{j}\{\psi(x)\}^{-(r+1)}}\right\}$$

$$\theta_{PLF}=\left\{\frac{(r+2)(r+1)\sum_{j=0}^{n-r}(-1)^{j}\binom{n-r}{j}\{\psi(x)\}^{-(r+3)}}{\sum_{j=0}^{n-r}(-1)^{j}\binom{n-r}{j}\{\psi(x)\}^{-(r+1)}}\right\}^{\frac{1}{2}}$$

The posterior risks for singly type II censored samples under uniform prior using SELF, QLF, WLF, LLF and PLF are:

$$\rho\left(\theta_{SELF}\right)=\frac{(r+2)(r+1)\sum_{j=0}^{n-r}(-1)^{j}\binom{n-r}{j}\{\psi(x)\}^{-(r+3)}}{\sum_{j=0}^{n-r}(-1)^{j}\binom{n-r}{j}\{\psi(x)\}^{-(r+1)}}-\left\{\frac{(r+1)\sum_{j=0}^{n-r}(-1)^{j}\binom{n-r}{j}\{\psi(x)\}^{-(r+2)}}{\sum_{j=0}^{n-r}(-1)^{j}\binom{n-r}{j}\{\psi(x)\}^{-(r+1)}}\right\}^{2}$$

$$\rho\left(\theta_{QLF}\right)=1-\frac{\left\{\sum_{j=0}^{n-r}(-1)^{j}\binom{n-r}{j}\{\psi(x)\}^{-r}\right\}^{2}}{\left[\sum_{j=0}^{n-r}(-1)^{j}\binom{n-r}{j}\{\psi(x)\}^{-(r+1)}\right]\left[\sum_{j=0}^{n-r}(-1)^{j}\binom{n-r}{j}\{\psi(x)\}^{-(r-1)}\right]}$$

$$\rho\left(\theta_{WLF}\right)=\frac{\left(r+1\right)\sum_{j=0}^{n-r}\left(-1\right)^{j}\binom{n-r}{j}\left\{\psi\left(x\right)\right\}^{-(r+2)}}{\sum_{j=0}^{n-r}\left(-1\right)^{j}\binom{n-r}{j}\left\{\psi\left(x\right)\right\}^{-(r+1)}}-\left\{\frac{\sum_{j=0}^{n-r}\left(-1\right)^{j}\binom{n-r}{j}\left\{\psi\left(x\right)\right\}^{-r}}{r\sum_{j=0}^{n-r}\left(-1\right)^{j}\binom{n-r}{j}\left\{\psi\left(x\right)\right\}^{-(r+1)}}\right\}^{-1}$$

$$\rho\left(\theta_{LLF}\right)=\frac{\left(r+1\right)\sum_{j=0}^{n-r}\left(-1\right)^{j}\binom{n-r}{j}\left\{\psi\left(x\right)\right\}^{-(r+2)}}{\sum_{j=0}^{n-r}\left(-1\right)^{j}\binom{n-r}{j}\left\{\psi\left(x\right)\right\}^{-(r+1)}}+\ln\left\{\frac{\sum_{j=0}^{n-r}\left(-1\right)^{j}\binom{n-r}{j}\left\{\psi\left(x\right)+1\right\}^{-(r+1)}}{\sum_{j=0}^{n-r}\left(-1\right)^{j}\binom{n-r}{j}\left\{\psi\left(x\right)\right\}^{-(r+1)}}\right\}$$

$$\rho\left(\theta_{PLF}\right)=2\left\{\frac{\left(r+2\right)\left(r+1\right)\sum_{j=0}^{n-r}\left(-1\right)^{j}\binom{n-r}{j}\left\{\psi\left(x\right)\right\}^{-(r+3)}}{\sum_{j=0}^{n-r}\left(-1\right)^{j}\binom{n-r}{j}\left\{\psi\left(x\right)\right\}^{-(r+1)}}\right\}^{\frac{1}{2}}-\frac{2\left(r+1\right)\sum_{j=0}^{n-r}\left(-1\right)^{j}\binom{n-r}{j}\left\{\psi\left(x\right)\right\}^{-(r+2)}}{\sum_{j=0}^{n-r}\left(-1\right)^{j}\binom{n-r}{j}\left\{\psi\left(x\right)\right\}^{-(r+1)}}$$

Bayesian Analysis for Doubly Type II Censored Samples

Consider a random sample of size ‘n’ from an Burr type XI distribution, and let $x_r,\ldots, x_s$ be the ordered observations remaining when the ‘r – 1’ smallest observations and the ‘n – s’ largest observations have been censored, The likelihood function for θ given the Type II doubly censored sample $\underline{x} = (x_r,\ldots, x_s)$, is:

$$L\left(\theta|\underline{x}\right)\propto\left[F\left(x_r|\theta\right)\right]^{r-1}\left[1-F\left(x_s|\theta\right)\right]^{n-s}\prod_{i=r}^{s}f\left(x_i|\theta\right) \tag{15}$$

$$L\left(\theta|\underline{x}\right)\propto\sum_{k=0}^{n-s}\left(-1\right)^{k}\binom{n-s}{k}\theta^{r}e^{-\theta\zeta(x)} \tag{16}$$

Where

$$\zeta\left(x\right)=\sum_{i=r}^{s}\ln\left\{x_i-\left(\frac{1}{2\pi}\right)Sin\left(2\pi x_i\right)\right\}^{-1}$$
$$+\left(r-1\right)\ln\left\{x_r-\left(\frac{1}{2\pi}\right)Sin\left(2\pi x_r\right)\right\}^{-1}+k\ln\left\{x_s-\left(\frac{1}{2\pi}\right)Sin\left(2\pi x_s\right)\right\}^{-1}$$

The posterior distribution for doubly type II censored samples assuming uniform prior is:

$$p\left(\theta|\underline{x}\right)=\frac{\sum_{k=0}^{n-s}\left(-1\right)^{k}\binom{n-s}{k}\theta^{r}e^{-\theta\zeta(x)}}{\Gamma\left(m+1\right)\sum_{k=0}^{n-s}\left(-1\right)^{k}\binom{n-s}{k}\left\{\zeta\left(x\right)\right\}^{-(m+1)}}\qquad,\theta>0 \tag{17}$$

The posterior distributions under remaining priors can be obtained as obtained in previous section.

The Bayes estimators under doubly type II censored samples based on uniform prior using SELF, QLF, WLF, LLF and PLF are:

$$\theta_{SELF} = \frac{(m+1)\sum_{j=0}^{n-r}(-1)^j \binom{n-r}{j}\{\zeta(x)\}^{-(m+2)}}{\sum_{j=0}^{n-r}(-1)^j \binom{n-r}{j}\{\zeta(x)\}^{-(m+1)}}$$

$$\theta_{QLF} = \frac{(m-1)\sum_{k=0}^{n-s}(-1)^k \binom{n-s}{k}\{\zeta(x)\}^{-m}}{\sum_{k=0}^{n-s}(-1)^k \binom{n-s}{k}\{\zeta(x)\}^{-(m-1)}} \quad \theta_{WLF} = \left\{\frac{\sum_{k=0}^{n-s}(-1)^k \binom{n-s}{k}\{\zeta(x)\}^{-m}}{m\sum_{k=0}^{n-s}(-1)^k \binom{n-s}{k}\{\zeta(x)\}^{-(m+1)}}\right\}^{-1}$$

$$\theta_{LLF} = -\ln\left\{\frac{\sum_{k=0}^{n-s}(-1)^k \binom{n-s}{k}\{\zeta(x)+1\}^{-(m+1)}}{\sum_{k=0}^{n-s}(-1)^k \binom{n-s}{k}\{\zeta(x)\}^{-(m+1)}}\right\}$$

$$\theta_{PLF} = \left\{\frac{(m+2)(m+1)\sum_{k=0}^{n-s}(-1)^k \binom{n-s}{k}\{\zeta(x)\}^{-(m+3)}}{\sum_{k=0}^{n-s}(-1)^k \binom{n-s}{k}\{\zeta(x)\}^{-(m+1)}}\right\}^{\frac{1}{2}}$$

The Bayes risks under doubly type II censored samples based on uniform prior using SELF, QLF, WLF, LLF and PLF are:

$$\rho(\theta_{SELF}) = \frac{(m+2)(m+1)\sum_{k=0}^{n-s}(-1)^k \binom{n-s}{k}\{\zeta(x)\}^{-(m+3)}}{\sum_{k=0}^{n-s}(-1)^k \binom{n-s}{k}\{\zeta(x)\}^{-(m+1)}} - \left\{\frac{(m+1)\sum_{k=0}^{n-s}(-1)^k \binom{n-s}{k}\{\zeta(x)\}^{-(m+2)}}{\sum_{k=0}^{n-s}(-1)^k \binom{n-s}{k}\{\zeta(x)\}^{-(m+1)}}\right\}^2$$

$$\rho(\theta_{QLF}) = 1 - \frac{\left\{\sum_{k=0}^{n-s}(-1)^k \binom{n-s}{k}\{\zeta(x)\}^{-m}\right\}^2}{\left[\sum_{k=0}^{n-s}(-1)^k \binom{n-s}{k}\{\zeta(x)\}^{-(m+1)}\right]\left[\sum_{k=0}^{n-s}(-1)^k \binom{n-s}{k}\{\zeta(x)\}^{-(m-1)}\right]}$$

$$\rho\left(\theta_{WLF}\right)=\frac{\left(m+1\right)\sum_{k=0}^{n-s}\left(-1\right)^{k}\binom{n-s}{k}\left\{\zeta\left(x\right)\right\}^{-\left(m+2\right)}}{\sum_{k=0}^{n-s}\left(-1\right)^{k}\binom{n-s}{k}\left\{\zeta\left(x\right)\right\}^{-\left(m+1\right)}}-\left\{\frac{\sum_{k=0}^{n-s}\left(-1\right)^{k}\binom{n-s}{k}\left\{\zeta\left(x\right)\right\}^{-m}}{m\sum_{k=0}^{n-s}\left(-1\right)^{k}\binom{n-s}{k}\left\{\zeta\left(x\right)\right\}^{-\left(m+1\right)}}\right\}^{-1}$$

$$\rho\left(\theta_{LLF}\right)=\frac{\left(m+1\right)\sum_{k=0}^{n-s}\left(-1\right)^{k}\binom{n-s}{k}\left\{\zeta\left(x\right)\right\}^{-\left(m+2\right)}}{\sum_{k=0}^{n-s}\left(-1\right)^{k}\binom{n-s}{k}\left\{\zeta\left(x\right)\right\}^{-\left(m+1\right)}}+\ln\left\{\frac{\sum_{k=0}^{n-s}\left(-1\right)^{k}\binom{n-s}{k}\left\{\zeta\left(x\right)+1\right\}^{-\left(m+1\right)}}{\sum_{k=0}^{n-s}\left(-1\right)^{k}\binom{n-s}{k}\left\{\zeta\left(x\right)\right\}^{-\left(m+1\right)}}\right\}$$

$$\rho\left(\theta_{PLF}\right)=2\left\{\frac{\left(m+2\right)\left(m+1\right)\sum_{k=0}^{n-s}\left(-1\right)^{k}\binom{n-s}{k}\left\{\zeta\left(x\right)\right\}^{-\left(m+3\right)}}{\sum_{k=0}^{n-s}\left(-1\right)^{k}\binom{n-s}{k}\left\{\zeta\left(x\right)\right\}^{-\left(m+1\right)}}\right\}^{\frac{1}{2}}-\frac{2\left(m+1\right)\sum_{k=0}^{n-s}\left(-1\right)^{k}\binom{n-s}{k}\left\{\zeta\left(x\right)\right\}^{-\left(m+2\right)}}{\sum_{k=0}^{n-s}\left(-1\right)^{k}\binom{n-s}{k}\left\{\zeta\left(x\right)\right\}^{-\left(m+1\right)}}$$

The Bayes estimators and corresponding posterior distributions can be derived in a similar manner.

Credible Intervals

The classical theory of confidence intervals for parameter estimates is not insightful; saying that 95% confidence interval means that if the repeated confidence intervals are constructed for different samples then 95% of them will contain the true value of the parameter. The particular confidence interval from any one sample may or may not contain the true parameter value. While, a 95% Bayesian credible interval contains the true parameter value with approximately 95% confidence. The credible interval is defined as: Let $\pi\left(\theta|\underline{x}\right)$ be the posterior distribution then a $100\left(1-\alpha\right)\%$ credible interval in any set C *is* such that $P_{\pi\left(\theta|\underline{x}\right)}\left(C\right)=1-\alpha$. The $100\left(1-\alpha\right)\%$ credible interval on the basis of uniform prior is given as:

$100\left(1-\alpha\right)\%$ credible interval (L,U) for type II singly censored samples under uniform prior is:

$$L:\frac{2\left\{\sum_{j=0}^{n-r}\left(-1\right)^{j}\binom{n-r}{j}\left\{\psi\left(x\right)\right\}^{-\left(r+2\right)}\right\}}{\chi^{2}_{\left(\alpha/2\right)\left\{2\left(r+2\right)\right\}}\left\{\sum_{j=0}^{n-r}\left(-1\right)^{j}\binom{n-r}{j}\left\{\psi\left(x\right)\right\}^{-\left(r+1\right)}\right\}}$$

$$U:\frac{2\left\{\sum_{j=0}^{n-r}\left(-1\right)^{j}\binom{n-r}{j}\left\{\psi\left(x\right)\right\}^{-\left(r+2\right)}\right\}}{\chi^{2}_{\left(1-\alpha/2\right)\left\{2\left(r+2\right)\right\}}\left\{\sum_{j=0}^{n-r}\left(-1\right)^{j}\binom{n-r}{j}\left\{\psi\left(x\right)\right\}^{-\left(r+1\right)}\right\}}$$

$100\left(1-\alpha\right)\%$ credible interval (L,U) for type II doubly censored samples under uniform prior is:

$$L:\frac{2\left\{\sum_{j=0}^{n-r}(-1)^{j}\binom{n-r}{j}\left\{\zeta\left(x\right)\right\}^{-(m+2)}\right\}}{\chi^{2}_{\left(\alpha/2\right)\left\{2(m+2)\right\}}\left\{\sum_{j=0}^{n-r}(-1)^{j}\binom{n-r}{j}\left\{\zeta\left(x\right)\right\}^{-(m+1)}\right\}};U:\frac{\left\{\sum_{j=0}^{n-r}(-1)^{j}\binom{n-r}{j}\left\{\zeta\left(x\right)\right\}^{-(m+2)}\right\}}{\chi^{2}_{\left(1-\alpha/2\right)\left\{2(m+2)\right\}}\left\{\sum_{j=0}^{n-r}(-1)^{j}\binom{n-r}{j}\left\{\zeta\left(x\right)\right\}^{-(m+1)}\right\}}$$

Interested reader may further refer to Tahir and Saleem (2011). $100\left(1-\alpha\right)\%$ credible intervals under Jeffreys, exponential, gamma and chi square priors based on type II singly and doubly censored samples can be constructed accordingly.

Posterior Predictive Distributions and Intervals

The posterior predictive distribution is used to make predictions of future observations, based on our best inferences on parameters determined through observations already made. Posterior predictive distribution can simply be obtained by the product of the posterior distribution and (conditional) independence (given the parameters) of the new observation from the "learning sample". It can be defined as:

$$p\left(y|\underline{x}\right)=\int_{0}^{\infty}p(\theta|\underline{x})\,f\left(y;\theta\right)d\theta \tag{18}$$

Where y = x_{n+1} be the future observation given the sample information $x = x_1, x_2, ..., x_n$, from of the model with unknown parameter θ.

The posterior predictive Interval can be obtained by solving the following two equations.

$$\int_{0}^{L}p\left(y|\underline{x}\right)dy=\frac{\alpha}{2},\qquad \int_{U}^{\infty}p\left(y|\underline{x}\right)dy=\frac{\alpha}{2}$$

The posterior predictive distribution under singly type II censored samples based on uniform prior is:

$$p\left(y|\underline{x}\right)=\frac{\sum_{j=0}^{n-r}(-1)^{j}\binom{n-r}{j}\left[\psi\left(x\right)+\left\{y-\left(\frac{1}{2\pi}\right)Sin\left(2\pi y\right)\right\}^{-1}\right]^{-(r+2)}}{\left(r+1\right)^{-1}\sum_{j=0}^{n-r}(-1)^{j}\binom{n-r}{j}\left\{\psi\left(x\right)\right\}^{-(r+1)}}\qquad y>0 \tag{19}$$

The posterior predictive interval for singly type II censored samples under the assumption of uniform prior can be obtained by solving the following two equations numerically.

$$\frac{\sum_{j=0}^{n-r}(-1)^j\binom{n-r}{j}\int_0^L\left[\psi(x)+\left\{y-\left(\frac{1}{2\pi}\right)Sin(2\pi y)\right\}^{-1}\right]^{-(r+2)}dy}{(r+1)^{-1}\sum_{j=0}^{n-r}(-1)^j\binom{n-r}{j}\{\psi(x)\}^{-(r+1)}}=\frac{\alpha}{2}$$

$$\frac{\sum_{j=0}^{n-r}(-1)^j\binom{n-r}{j}\int_U^\infty\left[\psi(x)+\left\{y-\left(\frac{1}{2\pi}\right)Sin(2\pi y)\right\}^{-1}\right]^{-(r+2)}dy}{(r+1)^{-1}\sum_{j=0}^{n-r}(-1)^j\binom{n-r}{j}\{\psi(x)\}^{-(r+1)}}=\frac{\alpha}{2}$$

The posterior predictive distribution under doubly type II censored samples based on uniform prior is:

$$p(y|\underline{x})=\frac{\sum_{k=0}^{n-s}(-1)^k\binom{n-s}{k}\left[\zeta(x)+\left\{y-\left(\frac{1}{2\pi}\right)Sin(2\pi y)\right\}^{-1}\right]^{-(m+2)}}{(m+1)^{-1}\sum_{k=0}^{n-s}(-1)^k\binom{n-s}{k}\{\zeta(x)\}^{-(m+1)}} \qquad y>0 \tag{20}$$

The posterior predictive interval for doubly type II censored samples under the assumption of uniform prior can be obtained by solving the following two equations numerically.

$$\frac{\sum_{k=0}^{n-s}(-1)^k\binom{n-s}{k}\int_0^L\left[\zeta(x)+\left\{y-\left(\frac{1}{2\pi}\right)Sin(2\pi y)\right\}^{-1}\right]^{-(m+2)}dy}{(m+1)^{-1}\sum_{k=0}^{n-s}(-1)^k\binom{n-s}{k}\{\zeta(x)\}^{-(m+1)}}=\frac{\alpha}{2}$$

$$\frac{\sum_{k=0}^{n-s}(-1)^k\binom{n-s}{k}\int_U^{¥}\left[\zeta(x)+\left\{y-\left(\frac{1}{2\pi}\right)Sin(2\pi y)\right\}^{-1}\right]^{-(m+2)}dy}{(m+1)^{-1}\sum_{k=0}^{n-s}(-1)^k\binom{n-s}{k}\{\zeta(x)\}^{-(m+1)}}=\frac{\alpha}{2}$$

With little modifications, the posterior predictive distributions and intervals can be constructed under Jeffreys, exponential, gamma and chi square priors.

3. Results and Discussions

The simulation study has been conducted for n = 100, 200 and 300 using different parametric values and various levels of censoring rates. While, the results are presented for θ = 4 under 20% singly and doubly censored samples. Similarly different values of the hyper-parameters have been assumed for estimation but the results have been presented for a = 1, b = 1.5, k = 2 and h = 2. However, the discussions have been made for all the simulated results in the coming sections. The amounts of posterior risks associated with each estimator have been given in the parenthesis in the tables.

Table 1. Bayes estimates and posterior risks under different priors and loss functions for 20% singly type II censored samples

Sample Size	Prior	Loss Functions SELF	QLF	WLF	LLF	PLF
100	Uniform	4.320498	4.151066	4.235782	4.111590	4.362650
		(0.598773)	(0.022000)	(0.113643)	(0.280243)	(0.113091)
200		4.222836	4.139216	4.181026	4.118704	4.243689
		(0.288837)	(0.011000)	(0.056087)	(0.139689)	(0.055949)
300		4.201680	4.146028	4.173854	4.131964	4.215570
		(0.191265)	(0.007334)	(0.037327)	(0.093522)	(0.037266)
100	Jeffreys	4.285934	4.117858	4.201896	4.078697	4.327749
		(0.568834)	(0.020900)	(0.107961)	(0.266230)	(0.107436)
200		4.189053	4.106102	4.147578	4.085755	4.209740
		(0.274395)	(0.010450)	(0.053282)	(0.132705)	(0.053152)
300		4.168067	4.112860	4.140463	4.098908	4.181845
		(0.181701)	(0.006967)	(0.035461)	(0.088846)	(0.035403)
100	Exponential	4.251647	4.084915	4.168281	4.046068	4.293127
		(0.540393)	(0.019855)	(0.102563)	(0.252919)	(0.102065)
200		4.155541	4.073253	4.114397	4.053069	4.176062
		(0.260675)	(0.009928)	(0.050618)	(0.126069)	(0.050494)
300		4.134722	4.079957	4.107340	4.066117	4.148391
		(0.172616)	(0.006619)	(0.033688)	(0.084404)	(0.033632)
100	Gamma	4.294163	4.125764	4.209964	4.086528	4.336058
		(0.497161)	(0.018267)	(0.094358)	(0.232685)	(0.093899)
200		4.197096	4.113986	4.155541	4.093599	4.217823
		(0.239821)	(0.009133)	(0.046569)	(0.115984)	(0.046455)
300		4.176069	4.120757	4.148413	4.106778	4.189875
		(0.158807)	(0.006089)	(0.030993)	(0.077651)	(0.030942)
100	Chi Square	4.281281	4.113387	4.197334	4.074269	4.323050
		(0.519036)	(0.019070)	(0.098510)	(0.242924)	(0.098031)
200		4.184505	4.101644	4.143074	4.081318	4.205169
		(0.250373)	(0.009535)	(0.048618)	(0.121087)	(0.048499)
300		4.163541	4.108394	4.135968	4.094457	4.177305
		(0.165795)	(0.006357)	(0.032357)	(0.081068)	(0.032303)

Table 2. Bayes estimates and posterior risks under different priors and loss functions for 20% doubly type II censored samples

Sample Size	Prior	Loss Functions				
		SELF	QLF	WLF	LLF	PLF
100	Uniform	4.383725 (0.625990)	4.211814 (0.023000)	4.297769 (0.118809)	4.171759 (0.292981)	4.426494 (0.118232)
200		4.284634 (0.301966)	4.199790 (0.011500)	4.242212 (0.058636)	4.178978 (0.146039)	4.305792 (0.058492)
300		4.263168 (0.199959)	4.206702 (0.007667)	4.234935 (0.039024)	4.192431 (0.097773)	4.277261 (0.038960)
100	Jeffreys	4.348655 (0.594690)	4.178119 (0.021850)	4.263387 (0.112868)	4.138385 (0.278332)	4.391082 (0.112320)
200		4.250357 (0.286868)	4.166191 (0.010925)	4.208274 (0.055704)	4.145546 (0.138737)	4.271346 (0.055568)
300		4.229063 (0.189961)	4.173048 (0.007284)	4.201055 (0.037073)	4.158892 (0.092884)	4.243043 (0.037012)
100	Exponential	4.313866 (0.564956)	4.144694 (0.020758)	4.229280 (0.107225)	4.105278 (0.264415)	4.355953 (0.106704)
200		4.216354 (0.272524)	4.132862 (0.010379)	4.174608 (0.052919)	4.112382 (0.131800)	4.237175 (0.052789)
300		4.195230 (0.180463)	4.139664 (0.006920)	4.167447 (0.035219)	4.125621 (0.088240)	4.209099 (0.035161)
100	Gamma	4.357004 (0.519759)	4.186141 (0.019097)	4.271573 (0.098647)	4.146331 (0.243262)	4.399513 (0.098168)
200		4.258517 (0.250722)	4.174190 (0.009548)	4.216354 (0.048686)	4.153506 (0.121256)	4.279547 (0.048566)
300		4.237182 (0.166026)	4.181060 (0.006366)	4.209121 (0.032402)	4.166877 (0.081181)	4.251190 (0.032348)
100	Chi Square	4.343933 (0.542629)	4.173583 (0.019937)	4.258758 (0.102987)	4.133892 (0.253966)	4.386314 (0.102487)
200		4.245742 (0.261754)	4.161668 (0.009969)	4.203705 (0.050828)	4.141045 (0.126591)	4.266708 (0.050703)
300		4.224471 (0.173331)	4.168517 (0.006646)	4.196494 (0.033827)	4.154376 (0.084753)	4.238436 (0.033772)

Table 3. 95% credible intervals under different priors based on 20% singly and doubly type II censored samples

Sample Size	Prior	Singly Type II Censored			Doubly Type II Censored		
		LL	UL	UL–LL	LL	UL	UL–LL
100	Uniform	3.476122	5.366091	1.889969	3.526992	5.444619	1.917627
200		3.617577	4.928213	1.310636	3.670517	5.000334	1.329816
300		3.711419	4.763877	1.052458	3.765732	4.833592	1.067860
100	Jeffreys	3.405741	5.257444	1.851703	3.455582	5.334382	1.878801
200		3.566347	4.858422	1.292076	3.618537	4.929521	1.310984
300		3.666450	4.706157	1.039706	3.720106	4.775027	1.054922
100	Exponential	3.420727	5.260412	1.839685	3.470786	5.337393	1.866607
200		3.559927	4.849677	1.289750	3.612024	4.920648	1.308624
300		3.652273	4.687960	1.035686	3.705721	4.756564	1.050843
100	Gamma	3.454934	5.272746	1.817812	3.505494	5.349908	1.844414
200		3.595527	4.862924	1.267398	3.648144	4.934089	1.285945
300		3.688796	4.712668	1.023872	3.742778	4.781634	1.038855
100	Chi Square	3.444569	5.297077	1.852508	3.494977	5.374595	1.879617
200		3.584740	4.883480	1.298739	3.637200	4.954945	1.317745
300		3.677730	4.720635	1.042905	3.731550	4.789717	1.058167

Sample Size	Prior	Singly Type II Censored			Doubly Type II Censored		
		LL	UL	UL–LL	LL	UL	UL–LL
100	Uniform	2.433286	17.17149	14.738205	2.398355	17.531673	15.133318
200		2.532304	15.770283	13.237979	2.495952	16.101074	13.605122
300		2.597993	15.244406	12.646413	2.560698	15.564166	13.003469
100	Jeffreys	2.384019	16.823821	14.439802	2.349795	17.176710	14.826915
200		2.496443	15.546952	13.050509	2.460605	15.873059	13.412453
300		2.566515	15.059702	12.493186	2.529672	15.375588	12.845916
100	Exponential	2.428716	16.780713	14.351997	2.394842	17.079659	14.684816
200		2.527548	15.470471	12.942922	2.492297	15.746074	13.253778
300		2.593114	14.954591	12.361477	2.556948	15.221004	12.664057
100	Gamma	2.314806	16.609148	14.294343	2.313626	16.852209	14.538583
200		2.409003	15.318211	12.909209	2.407775	15.542380	13.134605
300		2.471493	14.844905	12.373411	2.470234	15.062147	12.591913
100	Chi Square	2.307861	16.685792	14.377930	2.306685	17.037466	14.730781
200		2.401776	15.382961	12.981185	2.400552	15.707176	13.306624
300		2.464079	14.869999	12.405920	2.462823	15.183403	12.720580

The simulation study has explored some appealing properties of the Bayes estimators. It is interesting to note that by increasing the sample size the estimated value of the parameter converges to the true value of the parameter. The rate of convergence is higher for estimates under exponential prior based on LINEX loss function. The amount of overestimation has been seen in all the cases. This indicates that the corresponding posterior distributions are positively skewed. The extend of overestimation is greater in case of precautionary loss function. The bigger choices of hyper-parametric values increase the tendency of convergence but at the cost of inflated posterior risks. The rate of convergence is directly related to sample size, while it is inversely proportional to the censoring rate and true parametric values.

On the other hand, the magnitudes of posterior risks associated with each Bayes estimates tend to decrease as sample size increases. The increased censoring rate and true parametric values impose a negative impact on the performance of the estimates. The magnitudes of risks associated with informative priors are always smaller than those under non-informative priors. This simply proves the point that the informative priors are superior to the non-informative priors. The least amount of risks has been observed under gamma prior on the basis of quadratic loss function. Similarly, the performance of the singly type II censored samples seems better that doubly type II censored samples.

In case of Bayesian interval estimation it is found that widths of credible intervals decrease by increasing the sample size. The table 3 shows that the credible intervals under informative priors are pretty shorter than those under non-informative priors. It is interesting to note that each credible interval contains the corresponding true and estimated value of the parameter. All the credible intervals are skewed to right describing that the corresponding posterior distributions are positively skewed. The bigger values of the parameters and the higher degrees of censoring rate result in smaller levels of precisions. The most precise credible intervals have been observed under the assumption of gamma prior based on singly type II censored samples. So, the findings of the interval estimation are completely in accordance with those for point estimation.

The prediction of the future values of the variable is always of supreme importance in Bayesian and non-Bayesian framework. Under the Bayesian point of view, the posterior predictive intervals address this problem. Here the posterior predictive intervals based on the Burr type XI distribution have been evaluated numerically. The results suggest that in order to predict the future value from the said distribution, the assumption of gamma prior under the singly type II censored samples is the most reasonable. Hence, the results from the posterior predictions further strengthened the conclusions drawn from point and interval estimation.

4. Conclusions

The paper intend to investigate the behavior and performance of various point and interval estimators of the parameter of the Burr type XI distribution based on different priors and loss functions under singly and doubly type II censored samples. The framework for Bayesian predictions from the distribution has also been discussed. The study proposed the use of gamma prior along with quadratic loss function for Bayes estimation and prediction from the distribution. In addition, for the above analysis, the performance of the singly type II censoring scheme is found superior to the doubly type II censoring.

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