

# New Polyphase Complementary Sequence Sets for Wireless Communication Systems

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## Abstract

*In contemporary wireless communication systems, complementary sequences play fairly important roles. Based on polyphase perfect sequences (PPSs), this paper presents a construction method, whose basic idea is to sample a given PPS with equal space, for yielding a family of periodic polyphase complementary sequence sets (PPCSSs). The advantages of this method include the family size of resultant PPCSSs is the same as that of the PPSs employed, and the number and length of sub-sequences in the proposed PPCSSs can be altered on demand.*

**Keywords:** *complementary sequence, periodic sequence, polyphase sequence, perfect sequence*

## 1. Introduction

In contemporary wireless communications, there are two important communication systems. One of them is code-division multiple-access (CDMA) communication systems, and the other is orthogonal frequency-division multiplexing (OFDM) ones. However, there exists multiple-access interference (MAI) for the former, and high peak-to-average power ratio (PAPR) for the latter, which result in that the performance of communication systems degenerates. Hence, the methods for suppressing MAI are investigated [1-3], and the approaches for reducing PAPR are explored [4-5], therein complementary sequences play fairly important roles. The complementary sequence sets are widely applied to elimination of multiple access interference (MAI) [3], channel estimation [6-7], synchronization [8], reduction of PAPR [9], and so on. A complementary sequence set possesses an impulsive autocorrelation function, that is, the autocorrelation functions of all its sub-sequences sum to zero except the time shift in the center. Up to now, many construction methods for complementary sequence sets have been presented [10], in which Popović [11] proposed a construction method for aperiodic complementary sequence sets (ACSSs) by making use of polyphase perfect sequences (PPSs) and their cyclic time shifted versions, and the number and length of sub-sequences of the resulting ACSSs are equal to the ones of PPSs employed. In addition, Ref. [10] showed a construction for periodic complementary sequence sets (PCSSs) by using perfect arrays, and the obtained PCSSs possess that the number and length of their sub-sequences are the same as the numbers of rows and columns of perfect arrays employed, respectively. It is apparently disadvantageous that in two

methods referred to above, the number and length of sub-sequences don't be changed freely, which will result in application obstacle, such as in variable date-rate transmission. In this paper, a new construction method for periodic polyphase complementary sequence sets (PPCSSs) is presented by making use of sampling PPSs with equal space, and in the resulting PPCSSs, the number and length of sub-sequences can be altered according to requirements. As a consequence, such a difficulty mentioned above is conquered to some extent.

## 2. Basic Concepts

Let  $\underline{s}^{(i)} = (s_0^{(i)}, s_1^{(i)}, s_2^{(i)}, \dots, s_{M-1}^{(i)})$  denote a polyphase sequence with length  $M$ , whose elements take the unit magnitude, that is,  $|s_k^{(i)}|^2 = 1 (0 \leq k \leq M-1)$ . We define the periodic autocorrelation function of the sequence  $\underline{s}^{(i)}$  as following.

$$R_{s^{(i)}, s^{(i)}}(u) = \sum_{k=0}^{M-1} s_k^{(i)} (s_{k+u}^{(i)})^* \quad (|u| \leq M-1), \quad (1)$$

where  $(x)^*$  denotes the complex conjugate of  $x$ , and the subscript " $k+u$ " in (1) is operated modulo  $M$ .

Let sequence  $\underline{c}$  consist of  $N$  sub-sequences, more clearly,  $\underline{c} = (\underline{s}^{(0)}, \underline{s}^{(1)}, \underline{s}^{(2)}, \dots, \underline{s}^{(N-1)})$ . The sequence  $\underline{c}$  is referred to as a PPCSS if the periodic autocorrelation functions of all its sub-sequences sum to zero apart from time shift  $\tau = 0$ , namely,

$$R_{c,c}(\tau) = \sum_{i=0}^{N-1} R_{s^{(i)}, s^{(i)}}(\tau) = \begin{cases} NM & \tau \equiv 0 \pmod{M} \\ 0 & \tau \not\equiv 0 \pmod{M}. \end{cases} \quad (2)$$

Let polyphase sequence  $\underline{a} = (a_0, a_1, a_2, \dots, a_{L-1})$  have length  $L$ . If we have

$$R_{a,a}(u) = \sum_{k=0}^{L-1} a_k (a_{k+u})^* = \begin{cases} L & u \equiv 0 \pmod{L} \\ 0 & u \not\equiv 0 \pmod{L}, \end{cases} \quad (3)$$

we refer to the sequence  $\underline{a}$  as a perfect polyphase sequence (PPS).

## 3. Known PPSs

For the sake of convenience of the reader, in this section we will recall the several existing constructions of PPSs.

Construction 1: Zadoff-Chu construction [12].

Let  $L$  and  $\nu$  be two positive numbers that satisfy  $\gcd(\nu, L) = 1$ . We set

$$a_k = \begin{cases} e^{j\frac{\pi}{L}\nu k(k+1)} & \text{odd } L \\ e^{j\frac{\pi}{L}\nu k^2} & \text{even } L \end{cases} \quad (0 \leq k \leq L-1). \quad (4)$$

Then, the sequence  $\underline{a}_v = (a_0^{(v)}, a_1^{(v)}, a_2^{(v)}, \dots, a_{L-1}^{(v)})$  is referred to as a ZC sequence with root  $v$ .

**Lemma 1:** A ZC sequence is a PPS. And for the roots  $v_1$  and  $v_2$ , if both satisfy  $\gcd(v_1 - v_2, L) = 1$ , the crosscorrelation function of two ZC sequences from the roots  $v_1$  and  $v_2$  has

$$R_{a_{v_1}, a_{v_2}}(u) = \sqrt{L} \quad (\forall u), \quad (5)$$

which attains Welch lower bound.

Construction 2: Zeng construction [13].

This construction includes two steps as follows.

Step 1: Construct  $L$  polyphase sequences.

$$\begin{cases} b_l = (b_0^{(l)}, b_1^{(l)}, b_2^{(l)}, \dots, b_{L-1}^{(l)}) & (0 \leq l \leq L-1) \\ b_k^{(l)} = e^{j\frac{2\pi}{L}lk} & (0 \leq k \leq L-1). \end{cases} \quad (6)$$

Step 2: Yield an interleaved sequence.

Let  $l_0 l_1 l_2 \dots l_{L-1}$  denote an arbitrary permutation in the symbol set  $\{0, 1, 2, \dots, L-1\}$  and  $I[\cdot, \cdot, L, \cdot]$  stand for an interleaving operator (see [13] for more information). We obtain an interleaved sequence as follows.

$$a_{l_0 l_1 l_2 \dots l_{L-1}} = I[b_{l_0}, b_{l_1}, b_{l_2}, \dots, b_{l_{L-1}}]. \quad (7)$$

**Lemma 2:** The sequence  $a_{l_0 l_1 l_2 \dots l_{L-1}}$  in Eq. (7) is a PPS. And the number of distinct sequences in Zeng construction is  $(L-1) \cdot (L-1)!$ .

The performance of several existing constructions for producing PPSs is given in Table 1, where  $\phi(\cdot)$  stands for Euler's phi function.

**Table 1. The comparison of several existing constructions of PPSs**

family	length $L^2$	$3^2$	$4^2$	$5^2$	$6^2$	$7^2$	$8^2$
	family size						
ZC [12]	$\phi(L^2)$	6	12	20	30	42	56
Frank [14]	$\phi(L)$	2	2	4	2	6	4
Zeng [13]	$(L-1) \cdot (L-1)!$	4	18	96	600	4320	35280

### 4. New Construction of PPCSSs

In this section, a new construction for PPCSSs is presented. For arbitrary given positive integers  $N$  and  $M$ , we freely choose a PPS  $\underline{a}$  with length  $L = MN$  expressed by

$$\underline{a} = (a_\lambda, a_{\lambda+1}, \dots, a_{\lambda+N-1}, a_{\lambda+N}, a_{\lambda+N+1}, \dots, a_{\lambda+2N-1}, \dots, a_{\lambda+(M-1)N}, a_{\lambda+(M-1)N+1}, \dots, a_{\lambda+(M-1)N+N-1}), \tag{8}$$

where  $0 \leq \lambda \leq L-1$ , i.e., we consider the PPS in arbitrary a period.

Now, we carry out sampling the PPS  $\underline{a}$  in (8) with equal space  $N-1$ . More clearly, we assign an element with subscript  $\lambda+l$  ( $0 \leq l \leq N-1$ ) in (8) as an initial one, and extract those element whose subscripts are exactly  $\lambda+kN+l$  ( $0 \leq k \leq M-1$ ). Hence, by arranging all obtained elements in natural subscripts' order, a sub-sequence is yielded, which is denoted by  $\underline{b}^{(\lambda+l)_N}$  for convenience, where  $(x)_N$  denotes the residue of  $x$  modulo  $N$ . After  $l$  ranges in the range from 0 to  $N-1$ , a PPCSS is obtained, which consists of all the obtained sub-sequences. For the sake of understanding easy to the reader, the operation referred to above is visually described in Figure 1. In a mathematical term, we have the following  $N$  sub-sequences  $\underline{b}^{(\lambda+l)_N}$  ( $0 \leq l \leq N-1$ ).

$$\begin{aligned} \underline{b}^{(\lambda)_N} &= (b_0^{(\lambda)_N}, b_1^{(\lambda)_N}, b_2^{(\lambda)_N}, \dots, b_{M-1}^{(\lambda)_N}) \\ \underline{b}^{(\lambda+1)_N} &= (b_0^{(\lambda+1)_N}, b_1^{(\lambda+1)_N}, b_2^{(\lambda+1)_N}, \dots, b_{M-1}^{(\lambda+1)_N}) \\ &\vdots \\ \underline{b}^{(\lambda+N-1)_N} &= (b_0^{(\lambda+N-1)_N}, b_1^{(\lambda+N-1)_N}, b_2^{(\lambda+N-1)_N}, \dots, b_{M-1}^{(\lambda+N-1)_N}), \end{aligned} \tag{9}$$

where  $b_k^{(\lambda+i)_N} = a_{\lambda+kN+i}$  ( $i = 0, 1, 2, \dots, N-1; k = 0, 1, 2, \dots, M-1$ ).

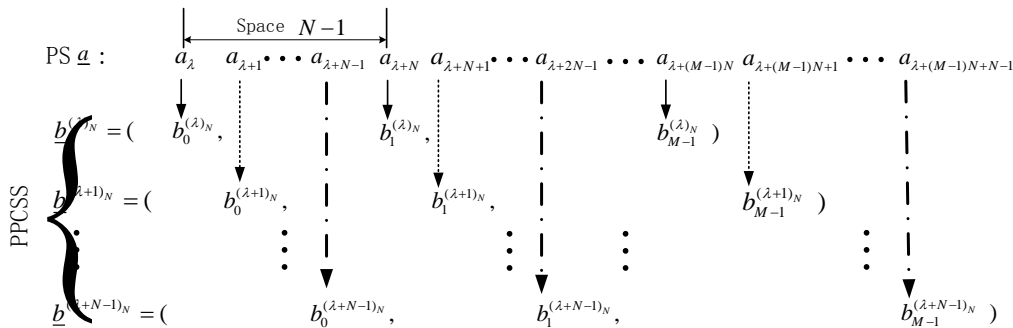


Figure 1. A description on new construction for PPCSSs

Note that the indices  $(\lambda+i)_N$  ( $i = 0, 1, 2, \dots, N-1$ ) of sub-sequences in (9) are given only for the sake of convenient expression in Figure 1, in fact, those indices can be freely assigned.

Then we declare the following conclusion.

**Theorem 1:** The sequence  $\underline{c} = (\underline{b}^{(\lambda)_N}, \underline{b}^{(\lambda+1)_N}, \underline{b}^{(\lambda+2)_N}, \dots, \underline{b}^{(\lambda+N-1)_N})$  is a PPCSS.

*Proof:* We consider the summation of the periodic autocorrelation functions of all subsequences in (9), therefore we have

$$\begin{aligned}
 R_{c,c}(\tau) &= \sum_{i=0}^{N-1} R_{\underline{b}^{(\lambda+i)_N}, \underline{b}^{(\lambda+i)_N}}(\tau) \quad (|\tau| \leq M-1) & (10) \\
 &= \sum_{i=0}^{N-1} \sum_{d=0}^{M-1} b_d^{(\lambda+i)_N} [b_{d+\tau}^{(\lambda+i)_N}]^* \\
 &= \sum_{i=0}^{N-1} \sum_{d=0}^{M-1} a_{\lambda+dN+i} [a_{\lambda+(d+\tau)N+i}]^* \\
 &= \sum_{k=0}^{L-1} a_{k+\lambda} (a_{k+\lambda+\tau N})^* \\
 &= R_{a,a}(\tau N).
 \end{aligned}$$

Note that the sequence  $\underline{a}$  in (8), employed by us, is a PPS, therefore its periodic autocorrelation function satisfies

$$R_{a,a}(u) = \sum_{k=0}^{L-1} a_{k+\lambda} (a_{k+\lambda+u})^* = \begin{cases} L & u \equiv 0 \pmod{L} \\ 0 & u \not\equiv 0 \pmod{L}. \end{cases} \quad (11)$$

Hence, due to  $|\tau N| \leq L-1$  ( $|\tau| \leq M-1$ ) and in accordance with (10) and (11), we obtain

$$R_{c,c}(\tau) = \begin{cases} L & \tau \equiv 0 \pmod{M} \\ 0 & \tau \not\equiv 0 \pmod{M}, \end{cases} \quad (12)$$

which illuminates that the theorem is true. We are done. ■

For a combination number  $L$ , it is possible that there are several factors. For instance,  $L=12=2 \times 6=3 \times 4$ . Apparently, for a given PPS  $\underline{a}$  with a period  $L$  of combination number, each kind of decomposition of  $L$  will results in a different PPCSS. For example, when  $L=12$ , from Theorem 1 there are four classes of the resulting PPCSSs with  $(M, N) = (2, 6), (3, 4), (4, 3)$ , and  $(6, 2)$ , respectively. Hence, we refer to all the resultant PPCSSs as a class of PPCSSs associated with  $\underline{a}$ . This motivates that we investigate how many classes of PPCSSs Theorem 1 can produce for a given PPS. For solving this problem, a following lemma is necessary.

**Lemma 3:** [15, 16] Let  $A(L, P)$  be the total number of PPSs with length  $L$  and alphabet size  $P$ . Let  $L = sm^2$ , where  $s$  and  $m$  are both positive integers with  $s$  square-free. Then

$$A(L, P) \geq \begin{cases} m!s^m \phi^m(s) P^m & P_{\min} \text{ divides } P \\ 0 & \text{otherwise,} \end{cases} \quad (13)$$

where

$$P_{\min} = \begin{cases} 2sm & \text{for } s \text{ even and } m \text{ odd} \\ sm & \text{otherwise.} \end{cases} \quad (14)$$

The following theorem answers our problem.

**Theorem 2:** Let the conditions be the same ones in Lemma 2. Then, Theorem 1 results in  $A(L, P)$  classes of PPCSSs.

According to Lemma 3 and Table 3.1 in [14], from Theorem 1 the number of classes of the resultant PPCSSs up to length  $L=12$  and  $P=10$  is given in Table 2.

**Table 2. Number of classes of PPCSSs from Theorem 1 up to  $L=12$  and  $P=15$**

$\begin{matrix} P \\ L \end{matrix}$	2	3	4	5	6	7	8	9	10	PPCSSs ( $M, N$ )'s
2			8				16			
3		8			36			54		
4	8		32		72		128		200	(2,2)
5				100					200	
6										(2,3),(3,2)
7						294				
8			128				512			(2,4),(4,2)
9		162			1296			4374		(3,3)
10										
11										
12					2592					(2,6),(3,4), (4,3),(6,2)

**Table 2. -continued.**

$\begin{matrix} P \\ L \end{matrix}$	11	12	13	14	15	PPCSSs ( $M, N$ )'s
2		24				
3		72			90	
4		288		392		(2,2)
5					300	
6				144		(2,3),(3,2)
7				588		
8		1152				(2,4),(4,2)
9		10368			20250	(3,3)
10						
11	1210					
12						(2,6),(3,4), (4,3),(6,2)

## 5. An Example and Discussion

To illuminate our method's validity, a simple example is considered for space limitation. Let  $N = 4$  and  $M = 8$ . A ZC sequence [12] with length  $L = 32$  is employed.

$$\underline{a} = (0, 1, 4, 9, 16, 25, 36, 49, 0, 17, 36, 57, 16, 41, 4, 33, 0, 33, 4, 41, 16, 57, 36, 17, 0, 49, 36, 25, 16, 9, 4, 1),$$

where the element " $e^{j\frac{\pi}{MN}x}$ " is denoted by " $x$ " for simplification. Set  $\lambda = 15$ . After making use of the new construction, we have a PPCSS  $\underline{c} = (\underline{b}^{(3)}, \underline{b}^{(0)}, \underline{b}^{(1)}, \underline{b}^{(2)})$ , where

$$\underline{b}^{(0)} = (0, 16, 0, 16, 0, 16, 0, 16),$$

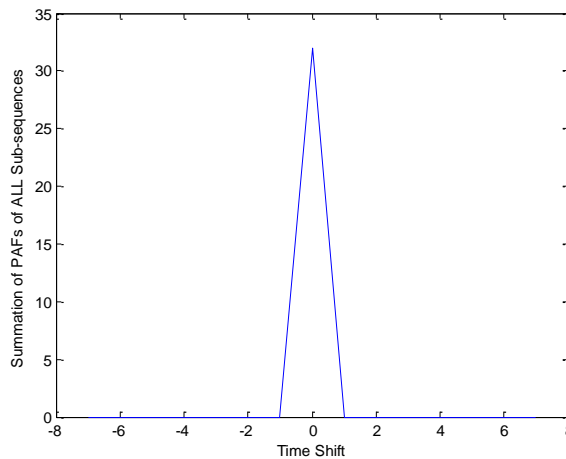
$$\underline{b}^{(1)} = (33, 57, 49, 9, 1, 25, 17, 41),$$

$$\underline{b}^{(2)} = (4, 36, 36, 4, 4, 36, 36, 4),$$

and

$$\underline{b}^{(3)} = (33, 41, 17, 25, 1, 9, 49, 57).$$

Figure 2 shows the summation of the periodic autocorrelation functions of 4 sub-sequences.



**Figure 2. Summation of periodic autocorrelation functions (PAFs) of all sub-sequences in the given example**

In known literature, there exist many methods that yield perfect sequences, such as Chu PSs (also known as Zadoff-Chu PSs) with length of arbitrary positive integer [12], Frank PSs with length of the square of a positive integer [14], modulatable orthogonal sequences with the same length as Frank PSs [17], and so on. Therefore, the perfect sequences required by the proposed construction are sufficiently rich so that for arbitrary given number and length of sub-sequences, at least a PPCSS can be obtained, which implies that the number and length of sub-sequences in the resulting PPCSSs can be altered on demand. On the other hand, for given positive integers  $N$  and  $M$ , a PPCSS can be produced by the proposed construction and a chosen PS with length  $L = MN$ , and a different PPCSS can be given by another employed PS with the same

length referred above, which implies the family size of PPCSSs obtained by this paper is equivalent to that of the employed PSs with length  $L = MN$ .

## 6. Conclusions

In this paper, a new construction which can produce PPCSSs is presented, and in the resulting PPCSSs, the number and length of sub-sequences can be changed on demand. In addition, the family size of the proposed PPCSSs is equivalent to that of the employed PSs. The advantage mentioned above highlights that new construction in this paper is very fit for various engineering applications. Incidentally, Theorem 1 can be applied to any perfect sequences, such as ternary perfect sequences[18], multilevel perfect sequences[19], QPSK+ perfect sequences[20], 8-QAM+ perfect sequences[21], and so on, so as to produce periodic ternary complementary sequence sets (CSSs), periodic multilevel CSSs, periodic QPSK+ CSSs, periodic 8-QAM+ CSSs, etc..

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## References

- [1] H. H. Chen, "The next generation CDMA technologies", John Wiley & Sons Ltd., New York, (2007).
- [2] T. Luo, G. Wu, S. Q. Li, Y. L. Guan and C. L. Law, "DS-SS-CDMA and MC-SS-CDMA with per-user MMSE frequency domain equalization", International Journal of Hybrid Information Technology, vol. 1, no. 3, (2008), pp. 11-20.
- [3] H. H. Chen, D. Hank, M. E. Maganaz and M. Guizani, "Design of next-generation CDMA using orthogonal complementary codes and offset stacked spreading", IEEE Wireless Communications, (2007), pp. 61-69.
- [4] Y. Rahmatallah and S. Mohan, "Peak-to-average power ratio reduction in OFDM systems: a survey and taxonomy", IEEE Communications Surveys & Tutorials. Preprint in IEEE website, (2013).
- [5] A. Sroy, R. Li, F. Z. Zeng and M. Fall, "A novel iterative clipping and filtering technique for PAPR reduction of OFDM signals: system using DCT/IDCT transform", International Journal of Future Generation Communication and Networking, vol. 6, no. 1, (2013), pp. 1-8.
- [6] P. Spasojević and C. N. Georgiades, "Complementary sequences for ISI channel estimation", IEEE Trans. on Inf. Theory, vol. 47, no. 3, (2001), pp. 1145-1152.
- [7] N. Q. Zhou and H. Liu, "Nonlinear channel estimation based on multi-level PN sequences in OFDM systems", International Journal of Hybrid Information Technology, vol. 1, no. 2, (2008), pp. 121-129.
- [8] D. Lowe and X. Huang, "Complementary channel estimation and synchronization for OFDM", Proceedings of the 2nd Int. Conf. on Wireless Broadband and Ultra Wideband Commun, Auswireless, (2007) August 27-30.
- [9] J. A. Davis and J. Jedwab, "Peak-to-Mean Power Control in OFDM, Golay complementary sequences, and Reed-Muller Codes", IEEE Trans. on Inf. Theory, vol. 45, no. 7, (1999), pp. 2397-2417.
- [10] P. Z. Fan and M. Darnell, "Sequence design for communications applications", John Wiley & Sons Inc., New York, (1996).
- [11] B. M. Popović, "Complementary sets based on sequences with ideal periodic autocorrelation", Electronics Letters, vol. 26, no. 18, (1990), pp. 1428-1430.
- [12] D. Chu, "Polyphase codes with good periodic correlation properties", IEEE Trans. On Inf. Theory, vol. 18, no. 4, (1972), pp. 531-532.
- [13] F. X. Zeng, "New perfect polyphase sequences and mutually orthogonal ZCZ polyphase sequence sets", IEICE Trans. Fundamentals, vol. E92-A, no. 7, (2009), pp. 1731-1736.
- [14] R. Frank, "Phase shift pulse codes with good periodic correlation properties", IRE Trans. on Inf. Theory, IT-8, (1962), pp. 381-382.
- [15] W. H. Mow, "Sequence design for spread spectrum", The Chinese University Press, Hong Kong (1997).



- [16] W. H. Mow, "A new unified construction of perfect root-of-unity sequence", Proceedings of IEEE 4th International Symposium on Spread Spectrum Techniques and Applications, Mainz, Germany, (1996) Spet. 22-25
- [17] N. Suehiro, *et al.*, "Modulatable orthogonal sequences and their application to SSMA systems", IEEE Trans. on Inf. Theory, vol. 34, no. 1, (1988), pp. 93-100.
- [18] T. Hoholdt and J. Justesen, "Ternary sequences with perfect periodic autocorrelation" IEEE Trans. on Inf. Theory, vol. 29, no. 4, (1983), pp. 597-600.
- [19] X. D. Li, P. Z. Fan, W. H. Mow and M. Darnell, "Multilevel perfect sequences over integers. Electronic. Lett., vol. 47, no. 8, (2011), pp. 496-497.
- [20] F. X. Zeng, X. P. Zeng, X. Y. Zeng, Z. Y. Zhang and G. X. Xuan, Several types of sequences with optimal autocorrelation properties. IEICE Trans. Fundamentals. E96-A, no. 1, (2013), pp. 367-372.
- [21] F. X. Zeng, X. P. Zeng, X. Y. Zeng, Z. Y. Zhang and G. X. Xuan, "Perfect 8-QAM+ sequences", IEEE Wireless Communications Letters, vol. 1, no. 4, (2012), pp. 388-391.

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