

# Approaches to Attribute Reduction in Concept Lattices Based on Rough Set Theory

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## **Abstract**

*This paper mainly proposes notions and methods of attribute reduction in concept lattices based on rough set theory. Using dependence space of concept lattices, we first discuss the relationships between congruence relations and the corresponding concept lattices. We then define notions of attribute reduction in a formal context based on congruence relations which is to find the minimal attribute subsets preserving the congruence partition. Finally, we define discernibility matrices and Boolean functions of a formal context to calculate all attribute reducts and analyze attribute characteristics. Using this notion of attribute reduction, methods, results as well as their proof about attribute reduction in a formal context can be derived directly by those in rough set theory. Furthermore, we prove that the attribute reducts proposed in this paper also preserve all extents of formal concepts and their original hierarchy in the concept lattice.*

**Keywords:** *Concept lattice, Formal context, Attribute reduction, Congruence relation, Rough set.*

## **1. Introduction**

Formal concept analysis [2, 23] and rough set theory [13, 14] are two effective tools for data analysis, knowledge representation and information management. The basic notions of formal concept analysis are formal contexts and concept lattices. A concept lattice is a partially ordered hierarchical structure of concepts that are defined by a binary relation between an object set and an attribute set. In rough set theory the lower and upper approximations of an arbitrary subset of universe are the basic operators. At present, the two theories both have many important applications in various fields respectively, and many efforts have been made to compare and combine the two theories.

Knowledge reduction is one of important problems in knowledge discovery and data mining. Recently, knowledge reduction in formal concept analysis and rough set theory both has achieved abundant research results. In formal concept analysis, Ganter et al. [2] proposed reducible attribute and reducible object from the viewpoint of shortening lines or rows. In [28], Zhang et al. presented an attribute reduction approach to find minimal attribute sets which can determine all extents and their original hierarchy in the concept lattice. And attribute reduction in a consistent formal decision context was also investigated in [22, 27]. Then, the approach was generalized to attribute reduction in the attribute oriented concept lattices and the object oriented concept lattices [9]. Wang et al. [19, 20] provided another approach to attribute reduction, which is only required preserving all extents of  $\wedge$ -irreducible elements. Wu et al. [24] studied attribute reduction in formal contexts from

the viewpoint of keeping granular structure of concept lattices. Liu et al. [8] showed an efficient post-processing method to prune redundant rules by virtue of the property of Galois connection, which inherently constrains rules with respect to objects. Mi et al. [10] formulated a Boolean approach to calculating all reducts of a formal context via the use of discernibility function. In [6], a rule acquisition oriented framework of knowledge reduction was proposed for real decision formal contexts and a corresponding reduction method was formulated by constructing a discernibility matrix and its associated Boolean function. Li and Zhang [7] introduced the notion of  $\delta$ -reducts in a fuzzy formal context, and give some equivalent characterizations of  $\delta$ -consistent sets to determine  $\delta$ -reducts. In [4], based on fuzzy K-means clustering, Kumar and Srinivas proposed a method to reduce the size of the concept lattices by employing corresponding object-attribute matrix.

In rough set theory, many types of attribute reductions have been proposed, each of the reductions aimed at some basic requirements [1, 3, 5, 11, 15, 16, 25, 26, 29]. In [11, 15], knowledge reduction in information systems is based on equivalence relation to find a minimal attribute set which can keep the partition of the object set of the information system. In [17], Skowron introduced the notion of discernibility matrix which became a major tool for searching for reductions in information systems. Then Zhang et al. [11, 26, 29] discussed approaches to attribute reduction in inconsistent and incomplete information systems using the similar idea. Kryszkiewicz [3] discussed relationships among other five kinds of knowledge reducts: generalized decision reducts, possible reduct, approximate reduct,  $\mu$ -reduct and  $\mu$ -decision reduct in inconsistent decision systems.

Relationships between attribute reducts in rough set theory and in the theory of concept lattices were discussed in [18, 21]. In [18], Wang and Zhang proved that an attribute reduct of a formal context [28] was also an attribute reduct in the sense of rough set theory [11, 15], just considering the formal context as an information system. In [21], the necessary and sufficient condition, which an attribute set was an attribute reduct in the sense of concept lattices and also in the sense of rough set theory, was given. What interests me is how to define notions of attribute reduction in concept lattices in the similar way with that in rough set theory? If this problem can be solved, then the corresponding results of knowledge reduction in formal concept analysis and rough set theory are interchangeable.

The main objective of this paper is to propose notions and approaches to attribute reduction in concept lattices which have the same expressions of attribute reduction in rough set theory. The paper is organized as follows. Section 2 recalls preliminaries on formal concept analysis, rough set theory and dependence space. Section 3 gives an approach to attribute reduction in a formal context based on congruence relations, which preserves the congruence partition of the formal context. Section 4 discusses the relationships between the approach in this paper and that in literature [28]. Finally, Section 5 concludes the paper.

## 2. Preliminaries

In this section, some basic notions and properties about formal concept analysis, rough set theory and dependence space are first introduced in this section.

### 2.1. Basic Notions about Formal Concept Analysis

**Definition 1** ([2]) *A formal context  $(U, A, I)$  consists of two sets  $U$  and  $A$ , and a relation  $I \subseteq U \times A$ . The elements of  $U$  are called objects and the elements of  $A$  are called attributes of the formal context.*

For  $X \subseteq U$  and  $B \subseteq A$ , Ganter and Wille defined two closure operators as follows:

$$X' = \{a \in A \mid \forall x \in X, (x, a) \in I\}, \quad (1)$$

$$B' = \{x \in U \mid \forall a \in B, (x, a) \in I\}. \quad (2)$$

**Definition 2** ([2]) Let  $(U, A, I)$  be a formal context. The formal context  $(U, B, I_B)$  is called a subcontext of  $(U, A, I)$ , where  $I_B = I \cap (U \times B)$  ( $B \subseteq A$ ).

Let 'B stand for the operator in the subcontext  $(U, B, I_B)$  ( $B \subseteq A$ ). Clearly, for  $X \subseteq U$ ,  $X^B = X^A \cap B$  and  $X^A = X'$ . For simplicity, we will write  $x'$  instead of  $\{x\}'$  and  $a'$  instead of  $\{a\}'$  for all  $(x, a) \in U \times A$ .

A formal context  $(U, A, I)$  is said to be canonical if the binary relation  $I$  is regular, that is, it satisfies the following conditions: for any  $(x, a) \in U \times A$ ,

(1) there exist  $a_1, a_2 \in A$  such that  $(x, a_1) \in I$  and  $(x, a_2) \notin I$ ,

(2) there exist  $x_1, x_2 \in U$  such that  $(x_1, a) \in I$  and  $(x_2, a) \notin I$ .

Alternatively, for each  $x \in U$ ,  $x' \neq \emptyset$  and  $x' \neq A$ , and for each  $a \in A$ ,  $a' \neq \emptyset$  and  $a' \neq U$ . We assume that all the formal contexts discussed in the paper are canonical.

**Definition 3** ([2]) A formal concept of a formal context  $(U, A, I)$  is a pair  $(X, B)$  with  $X \subseteq U, B \subseteq A, X' = B$  and  $B' = X$ . We call  $X$  the extent and  $B$  the intent of the concept  $(X, B)$ .

The concepts of a formal context  $(U, A, I)$  are partially ordered by

$$(X_1, B_1) \leq (X_2, B_2) \text{ if and only if (iff for short) } X_1 \subseteq X_2 \text{ (iff } B_2 \subseteq B_1)$$

where  $(X_1, B_1)$  and  $(X_2, B_2)$  are two concepts. The set of all concepts of  $(U, A, I)$  partially ordered in this way is denoted by  $L(U, A, I)$  and is called the concept lattice of the formal context  $(U, A, I)$ . The infimum and supremum are given by:

$$(X_1, B_1) \wedge (X_2, B_2) = (X_1 \cap X_2, (B_1 \cup B_2)''), \quad (3)$$

$$(X_1, B_1) \vee (X_2, B_2) = ((X_1 \cup X_2)', B_1 \cap B_2). \quad (4)$$

The concept lattice  $L(U, A, I)$  is a complete lattice. We denote the extent set of  $(U, A, I)$  by  $L_U(U, A, I) = \{X \mid (X, B) \in L(U, A, I)\}$ . It is evident that

$$L_U(U, D, I_D) \subseteq L_U(U, A, I) \text{ (} D \subseteq A \text{)}.$$

**Proposition 1** ([2]) Let  $(U, A, I)$  be a formal context,  $X, X_1, X_2$  be object sets, and  $B, B_1, B_2$  be attribute sets, then

$$\begin{array}{ll} (1) X_1 \subseteq X_2 \Rightarrow X_2' \subseteq X_1', & B_1 \subseteq B_2 \Rightarrow B_2' \subseteq B_1'; \\ (2) X \subseteq X'', & B \subseteq B''; \\ (3) X' = X''', & B' = B'''; \\ (4) (X_1 \cup X_2)' = X_1' \cap X_2', & (B_1 \cup B_2)' = B_1' \cap B_2'; \\ (5) (X_1 \cap X_2)' \supseteq X_1' \cup X_2', & (B_1 \cap B_2)' \supseteq B_1' \cup B_2'; \\ (6) (X'', X') \in L(U, A, I), & (B', B'') \in L(U, A, I). \end{array}$$

**Definition 4** ([27]) Let  $(U, A, I)$  be a formal context. If there exists an attribute set  $D \subseteq A$  such that  $L_U(U, D, I_D) = L_U(U, A, I)$ , then  $D$  is called a consistent set of  $(U, A, I)$ . And further, if  $D$  is a consistent set and no proper subset of  $D$  is consistent, then  $D$  is referred to as an attribute reduct of  $(U, A, I)$ .

Definition 4 shows that a consistent set in a concept lattice preserves all extents and their original hierarchy in the concept lattice.

**Definition 5** ([27]) Let  $(U, A, I)$  be a formal context,  $B_k (k \leq l)$  is an attribute reduct of  $(U, A, I)$ . Then the attributes are classified into the following three types:

- (1) redundant attribute  $b : b \in A - \bigcup_{k \leq l} B_k$ .
- (2) relative necessary attribute  $c : c \in \bigcup_{k \leq l} B_k - \bigcap_{k \leq l} B_k$ .
- (3) core attribute  $d : d \in \bigcap_{k \leq l} B_k$ .

## 2.2. Basic Notions about Rough Set Theory

**Definition 6** ([14]) An information system is a triple  $(U, A, F)$ , where  $U = \{x_1, x_2, \dots, x_n\}$  is a nonempty finite set of objects and  $A = \{a_1, a_2, \dots, a_m\}$  is a nonempty finite set of attributes,  $F$  is a set of functions between  $U$  and  $A$ , i.e.  $F = \{a_i | U \rightarrow V_i, a_i \rightarrow A\}$ ,  $V_i$  is the domain of  $a_i$ .

Comparing Definitions 1 and 6, a formal context can be taken as a two-valued information system. That is to say, a formal context is a particular information system.

Every nonempty subset  $B \subseteq A$  determines an equivalence relation as follows:

$$R_B = \{(x_i, x_j) \in U \times U \mid a_i(x_i) = a_i(x_j), \forall a_i \in B\}.$$

Since  $R_B$  is an equivalence relation on  $U$ , it partitions  $U$  into a family of disjoint subsets  $U / R_B$  called a quotient set of  $U, U / R_B = \{[x_i]_B \mid x_i \in U\}$ , where  $[x_i]_B$  denotes the equivalence class determined by  $x_i$  with respect to  $B$ , i.e.,  $[x_i]_B = \{x_j \in U \mid (x_i, x_j) \in R_B\}$ .

**Definition 7** ([27]) Let  $(U, A, F)$  be an information system and  $B \subseteq A$ . Then  $B$  is referred to as an attribute consistent set of  $(U, A, F)$  if  $R_B = R_A$ . And further, if  $B$  is a consistent set and no proper subset of  $B$  is consistent, then  $B$  is referred to as an attribute reduct of  $(U, A, F)$ .

Definition 7 says the consistent set of an information system preserves the equivalence partition of the object set of the information system.

## 2.3. Dependence Space based on a Formal Context

In [12], Novotný defined a congruence relation on the attribute power set  $P(A)$  and dependence space in information systems.

**Definition 8** ([12]) Let  $(U, A, F)$  be an information system.  $\mathbb{K}$  is an equivalence relation on  $\mathbb{P}(A)$ . Then,  $\mathbb{K}$  is called a congruence relation on  $(\mathbb{P}(A), \cup)$ , whenever it satisfies the following condition: if  $(B_1, C_1) \in \mathbb{K}, (B_2, C_2) \in \mathbb{K}$ , then  $(B_1 \cup B_2, C_1 \cup C_2) \in \mathbb{K}$ .

**Definition 9** ([12]) Let  $A$  be a finite nonempty set,  $\mathbb{K}$  a congruence relation on  $(\mathbb{P}(A), \cup)$ . Then the ordered pair  $(A, \mathbb{K})$  is said to be a dependence space.

### 3. Notions of Attribute Reduction in a Formal Context

Let  $(U, A, I)$  be a formal context. For  $B \subseteq A$ , we define a binary relation on the object power set  $\mathbb{P}(U)$  as follows:

$$R^B = \{(X, Y) \in \mathbb{P}(U) \times \mathbb{P}(U) \mid X^B = Y^B\}. \quad (5)$$

It is obvious that  $R^B$  is a congruence relation on  $(\mathbb{P}(U), \cup)$  and  $(U, R^B)$  is a dependence space.

#### 3.1. Relationships between Congruence Relations and the Corresponding Concept Lattices

We define  $[X]_{R^B} = \{Y \in \mathbb{P}(U) \mid (X, Y) \in R^B\}$ , the congruence class determined by  $X$  with respect to the congruence relation  $R^B$ , and  $C_{R^B}(X) = \cup\{Y \mid Y \in [X]_{R^B}\}$ . It is easy to verify the following result:

**Lemma 1** Let  $(U, A, I)$  be a formal context. For  $X, Y, Z \in \mathbb{P}(U)$  and  $B \subseteq A$ , the following statements hold:

- (1)  $(C_{R^B}(X), X) \in R^B$ .
- (2)  $C_{R^B}$  is a closure operator.
- (3) If  $X \subseteq Y \subseteq Z$  and  $(X, Z) \in R^B$ , then  $(X, Y) \in R^B$  and  $(Y, Z) \in R^B$ .

Lemma 1 shows that  $C_{R^B}(X)$  is the maximum element in  $[X]_{R^B}$ . An object subset  $X \in \mathbb{P}(U)$  is called  $C_{R^B}$ -closed if  $C_{R^B}(X) = X$ . The set of all  $C_{R^B}$ -closed sets is denoted by  $H_B$ .

**Lemma 2** Let  $(U, A, I)$  be a formal context. For  $X \in \mathbb{P}(U)$  and  $B \subseteq A$ , we have

- (1)  $C_{R^B}(X) = X^{B'B}$ .
- (2)  $H_B = L_U(U, B, I_B)$ .
- (3)  $(C_{R^B}(X), X^{B'}) \in L(U, B, I_B)$ .

*Proof.* (1) By Proposition 1 and Lemma 1, we have  $C_{R^B}(X) \subseteq X^{B'B}$ . Conversely, for any  $x \in X^{B'B}$ , clearly  $x^B \supseteq X^B$ . Let  $Y = X \cup \{x\}$ . Evidently,  $X^B = Y^B$ . Then,  $Y \subseteq C_{R^B}(X)$ . Notice that  $x \in Y$ , we conclude  $X^{B'B} \subseteq C_{R^B}(X)$ . Therefore,  $C_{R^B}(X) = X^{B'B}$ .

(2) For any  $X \in L_U(U, B, I_B)$ , it is clear that  $X = X^{B^B}$ . Then, by (1), we have  $X = C_{R^B}(X)$ , i.e.,  $X$  is a  $C_{R^B}$ -closed set. Therefore,  $L_U(U, B, I_B) \subseteq H_B$ . Furthermore, using (1) again we conclude  $H_B \subseteq L_U(U, B, I_B)$ .

(3) follows immediately from (1) and Lemma 1.

Lemma 2 says that all the  $C_{R^B}$ -closed sets form the extent set of  $(U, B, I_B)$  exactly.

**Lemma 3** Let  $(U, A_1, I_1)$  and  $(U, A_2, I_2)$  be two formal contexts with the same object set. If  $L_U(U, A_2, I_2) \subseteq L_U(U, A_1, I_1)$ , for  $X \in P(U)$ , we have

$$(1) C_{R^{A_1}}(C_{R^{A_2}}(X)) = C_{R^{A_2}}(X),$$

$$(2) C_{R^{A_1}}(X) \subseteq C_{R^{A_2}}(X).$$

*Proof.* (1) Since  $L_U(U, A_2, I_2) \subseteq L_U(U, A_1, I_1)$  and for  $X \in P(U)$ ,  $C_{R^{A_2}}(X) \in L_U(U, A_2, I_2)$  by Lemma 2, we have  $C_{R^{A_2}}(X) \in L_U(U, A_1, I_1)$  and  $(C_{R^{A_2}}(X))^{A_1 A_1} = C_{R^{A_2}}(X)$ . Combined with  $C_{R^{A_1}}(C_{R^{A_2}}(X)) = (C_{R^{A_2}}(X))^{A_1 A_1}$ , (i) is concluded.

(2) Since  $C_{R^{A_1}}$  is a closure operator, we have  $C_{R^{A_1}}(X) \subseteq C_{R^{A_1}}(C_{R^{A_2}}(X))$ . Thus,  $C_{R^{A_1}}(X) \subseteq C_{R^{A_2}}(X)$  follows directly from (i).

**Theorem 1** Let  $(U, A_1, I_1)$  and  $(U, A_2, I_2)$  be two formal contexts with the same object set. Then we have,  $L_U(U, A_2, I_2) \subseteq L_U(U, A_1, I_1) \Leftrightarrow R^{A_1} \subseteq R^{A_2}$ .

*Proof* Sufficiency. Assume  $L_U(U, A_2, I_2) \not\subseteq L_U(U, A_1, I_1)$ , then there exists  $X \in L_U(U, A_2, I_2)$  such that  $X \notin L_U(U, A_1, I_1)$ . Thus,  $C_{R^{A_2}}(X) \not\subseteq C_{R^{A_1}}(X)$  is concluded by Lemma 1. Since  $R^{A_1} \subseteq R^{A_2}$  implies  $[X]_{R^{A_1}} \subseteq [X]_{R^{A_2}}$ , we have  $C_{R^{A_1}}(X) \subseteq C_{R^{A_2}}(X)$ , which is a contradiction to  $C_{R^{A_2}}(X) \not\subseteq C_{R^{A_1}}(X)$ . Consequently,

$$L_U(U, A_2, I_2) \subseteq L_U(U, A_1, I_1).$$

Necessity. Assume  $R^{A_1} \not\subseteq R^{A_2}$ , then there exists  $X \in P(U)$  such that  $[X]_{R^{A_1}} \not\subseteq [X]_{R^{A_2}}$ . Thus, there exists  $Y \in [X]_{R^{A_1}}$  such that  $Y \notin [X]_{R^{A_2}}$ . We prove it from two cases:  $X \in L_U(U, A_1, I_1)$  and  $X \notin L_U(U, A_1, I_1)$ .

Firstly, we suppose  $X \in L_U(U, A_1, I_1)$ . Since  $Y \in [X]_{R^{A_1}}$  and  $Y \notin [X]_{R^{A_2}}$ , we obtain  $Y \not\subseteq X = C_{R^{A_1}}(Y)$ . Combining with  $C_{R^{A_1}}(Y) \subseteq C_{R^{A_2}}(Y)$  by Lemma 3, we have  $Y \not\subseteq X \subseteq C_{R^{A_2}}(Y)$ . Due to Lemma 1,  $(Y, X) \in R^{A_2}$ , which is a contradiction to  $Y \notin [X]_{R^{A_2}}$ . Therefore,  $[X]_{R^{A_1}} \subseteq [X]_{R^{A_2}}$  holds.

Secondly, we suppose  $X \notin L_U(U, A_1, I_1)$ . According to the above discussions, we have  $[C_{R^{A_1}}(X)]_{R^{A_1}} \subseteq [C_{R^{A_1}}(X)]_{R^{A_2}}$  due to  $C_{R^{A_1}}(X) \in L_U(U, A_1, I_1)$ . Since  $Y \in [X]_{R^{A_1}}$ , it is evident that  $Y \subseteq C_{R^{A_1}}(X)$  and  $Y \in [C_{R^{A_1}}(X)]_{R^{A_2}}$ . Combining with  $C_{R^{A_1}}(X) \subseteq C_{R^{A_2}}(X)$ , we have  $Y \subseteq C_{R^{A_2}}(X)$ . Since  $C_{R^{A_2}}$  is a closure operator and  $Y \in [C_{R^{A_1}}(X)]_{R^{A_2}}$ , we obtain  $C_{R^{A_2}}(X) \subseteq C_{R^{A_2}}(C_{R^{A_1}}(X)) = C_{R^{A_2}}(Y)$ . Thus,  $Y \subseteq C_{R^{A_2}}(X) \subseteq C_{R^{A_2}}(Y)$ . By Lemma 1,  $(Y, C_{R^{A_2}}(X)) \in R^{A_2}$  holds. That is,  $(Y, X) \in R^{A_2}$ , which is a contradiction to  $Y \notin [X]_{R^{A_2}}$ . Therefore,  $[X]_{R^{A_1}} \subseteq [X]_{R^{A_2}}$  is concluded.

Consequently, for  $X \subseteq U$ ,  $[X]_{R^{A_1}} \subseteq [X]_{R^{A_2}}$  holds.

### 3.2. Notions of Attribute Reduction in a Formal Context

Using the similar idea of attribute reduction in rough set theory, we give the following definition in a formal context based on congruence relations.

**Definition 10** Let  $(U, A, I)$  be a formal context. For  $D \subseteq A$ , if  $R^A = R^D$ , then  $D$  is called a consistent set of  $(U, A, I)$ . And further, if  $D$  is a consistent set and no proper subset of  $D$  is consistent, then  $D$  is referred to as an attribute reduct of  $(U, A, I)$ .

Definition 10 shows that the consistent sets of a formal context preserve all original congruence partition of the object set of the formal context. Comparing Definitions 7 and 10, the expressions of attribute reduction in concept lattices and rough set theory are very much the same. In the next section, we will show that approaches to attribute reduction in a formal context can be derived with the similar way used in rough set theory.

**Theorem 2** Let  $(U, A, I)$  be a formal context. For  $B \subseteq A$ , we have

- (1)  $B$  is a consistent set iff  $L_U(U, A, I) = L_U(U, B, I_B)$ .
- (2)  $B$  is an attribute reduct iff  $L_U(U, A, I) = L_U(U, B, I_B)$ , and for  $b \in B$ 

$$L_U(U, A, I) \neq L_U(U, B - \{b\}, I_{B - \{b\}}).$$

*Proof.* (1) By Definition 10 and Theorem 1, it is easy to observe that

$$\begin{aligned}
 B \text{ is a consistent set} & \text{ iff } R^A = R^B \\
 & \text{ iff } R^B \subseteq R^A \\
 & \text{ iff } L_U(U, A, I) \subseteq L_U(U, B, I_B) \\
 & \text{ iff } L_U(U, A, I) = L_U(U, B, I_B).
 \end{aligned}$$

- (2) The result follows directly from (i) and Definition 10.

*Remark.* Theorem 2 shows that the approach to attribute reduction is to find the minimal attribute sets which can also preserve all the extents and their original hierarchy in the formal context. Therefore, comparing with Definition 4 the approach proposed in this paper is in essence equivalent to the approach in [28]. However, these two approaches study reduction theory from different aspects. Literature [28] studies reduction theory from the isomorphism of the concept lattices, whereas, the method in this paper is from the viewpoint of preserving congruence classes.

#### 4. Approaches to Attribute Reduction in a Formal Context

In this section, we will define discernibility matrices and functions to calculate all attribute reducts and analyze attribute characteristics.

For convenience, we use  $R^a$  instead of  $R^{(a)}$  ( $a \in A$ ).

**Definition 11** Let  $(U, A, I)$  be a formal context. For  $X_i, X_j \in \mathbb{P}(U)$ , we define

$$D([X_i]_{R^A}, [X_j]_{R^A}) = \{a \in A \mid (X_i, X_j) \notin R^a\}. \quad (6)$$

Then  $D([X_i]_{R^A}, [X_j]_{R^A})$  is called the discernibility attribute set between  $[X_i]_{R^A}$  and  $[X_j]_{R^A}$  based on  $(U, A, I)$ , and  $D = (D([X_i]_{R^A}, [X_j]_{R^A}) \mid X_i, X_j \in \mathbb{P}(U))$  is called the discernibility matrix of  $(U, A, I)$ .

**Theorem 3** Let  $(U, A, I)$  be a formal context. For  $X_i, X_j \in \mathbb{P}(U)$ , the following statement holds:

$$D([X_i]_{R^A}, [X_j]_{R^A}) = B_i \cup B_j - B_i \cap B_j,$$

where  $(C_{R^A}(X_i), B_i) \in L(U, A, I)$  and  $(C_{R^A}(X_j), B_j) \in L(U, A, I)$ .

*Proof.* According to formulas (5) and (6),  $(C_{R^A}(X_i), B_i) \in L(U, A, I)$  and  $(C_{R^A}(X_j), B_j) \in L(U, A, I)$  for  $X_i, X_j \in \mathbb{P}(U)$ , we obtain

$$\begin{aligned} a \in D([X_i]_{R^A}, [X_j]_{R^A}) &\text{ iff } (X_i, X_j) \notin R^a \\ &\text{ iff } X_i^a \neq X_j^a \\ &\text{ iff } (C_{R^A}(X_i))^a \neq (C_{R^A}(X_j))^a \\ &\text{ iff } a \in B_i \cup B_j - B_i \cap B_j. \end{aligned}$$

By Definition 11, the discernibility attribute sets have the following properties:

**Proposition 2** Let  $(U, A, I)$  be a formal context. For  $X_i, X_j, X_k \in \mathbb{P}(U)$ , the following properties hold:

- (1)  $D([X_i]_{R^A}, [X_i]_{R^A}) = \emptyset$ .
- (2)  $D([X_i]_{R^A}, [X_j]_{R^A}) = D([X_j]_{R^A}, [X_i]_{R^A})$ .
- (3)  $D([X_i]_{R^A}, [X_j]_{R^A}) \subseteq D([X_i]_{R^A}, [X_k]_{R^A}) \cup D([X_k]_{R^A}, [X_j]_{R^A})$ .

Proposition 2 says the discernibility matrix of  $(U, A, I)$  is symmetric and the elements in the main diagonal are all empty sets.

**Theorem 4** Let  $(U, A, I)$  be a formal context. For  $B \subseteq A$ , we have

$B$  is a consistent set iff if  $D([X_i]_{R^A}, [X_j]_{R^A}) \neq \emptyset$ , then  $B \cap D([X_i]_{R^A}, [X_j]_{R^A}) \neq \emptyset$ ,



for all  $X_i, X_j \in P(U)$ .

**Theorem 5** Let  $(U, A, I)$  be a formal context and  $a \in A$ . The following statements are equivalent:

- (1)  $a$  is a core attribute,
- (2)  $R^{A-\{a\}} \neq R^A$ ,
- (3) there exist  $X_i, X_j \in P(U)$  such that  $D([X_i]_{R^A}, [X_j]_{R^A}) = \{a\}$ .

*Proof.* (1)  $\Rightarrow$  (2). Assume  $R^{A-\{a\}} = R^A$ . Then  $A - \{a\}$  is a consistent set by Definition 6. Thus, there exists  $B \subseteq A - \{a\}$  such that  $B$  is an attribute reduct and  $a \notin B$ , which is a contradiction to that  $a$  is a core attribute. Therefore, the result follows immediately.

(2)  $\Rightarrow$  (3). Since  $R^{A-\{a\}} \neq R^A$ , there must exist  $X_i, X_j \in P(U)$  such that  $(X_i, X_j) \notin R^A$  and  $(X_i, X_j) \in R^{A-\{a\}}$ . In other words,  $(X_i, X_j) \notin R^a$  and  $(X_i, X_j) \in R^b$ , for all  $b \in A$  and  $b \neq a$ . Therefore,  $D([X_i]_{R^A}, [X_j]_{R^A}) = \{a\}$  holds.

(3)  $\Rightarrow$  (1) can be directly concluded from Theorem 4.

$$\begin{aligned} \text{Denoting } R(a) &= \{(X, Y) \in P(U) \times P(U) \mid (X, Y) \in R^{B-\{a\}}, \text{ and } B \text{ is consistent}\} \\ &= \bigcup \{R^{B-\{a\}} \mid R^B \subseteq R^A, B \subseteq A\}, \end{aligned}$$

then we have the following result:

**Theorem 6** Let  $(U, A, I)$  be a formal context and  $a \in A$ . Then,  $a$  is a redundant attribute iff  $R(a) \subseteq R^a$ .

*Proof.* Necessity. If  $a$  is a redundant attribute, then for any attribute reduct  $B$ ,  $a \notin B$ . We observe that for  $B \subseteq A$ , if  $R^B \subseteq R^A$ , then  $R^{B-\{a\}} \subseteq R^A$ . Otherwise, if  $R^{B-\{a\}} \not\subseteq R^A$ , then for  $E \subseteq B - \{a\}$ ,  $R^E \not\subseteq R^A$ . Thus,  $B$  is an attribute reduct and  $a \in B$ , which is a contradiction to that  $a$  is a redundant attribute. Therefore,  $R(a) \subseteq R^A \subseteq R^a$  is concluded.

Sufficiency. If  $R(a) \subseteq R^a$ , then for  $B \subseteq A$ ,  $R^B \subseteq R^A$  implies  $R^{B-\{a\}} \subseteq R^a$ . Thus,  $R^{B-\{a\}} = R^{B-\{a\}} \cap R^a = R^B \subseteq R^A$ . Consequently, if  $B$  is a consistent set, then  $B - \{a\}$  is also a consistent set. That is,  $a$  is a redundant attribute.  $\square$

From Theorems 5 and 6, we can obtain the following theorem directly:

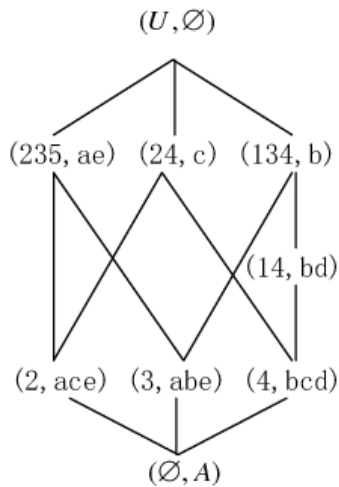
**Theorem 7** Let  $(U, A, I)$  be a formal context and  $a \in A$ . The following statements hold:

- (1)  $a$  is a core attribute iff  $R^{A-\{a\}} \neq R^A$ ,
- (2)  $a$  is a relative necessary attribute iff  $R^{A-\{a\}} = R^A$  and  $R(a) \not\subseteq R^a$ ,
- (3)  $a$  is a redundant attribute iff  $R(a) \subseteq R^a$ .

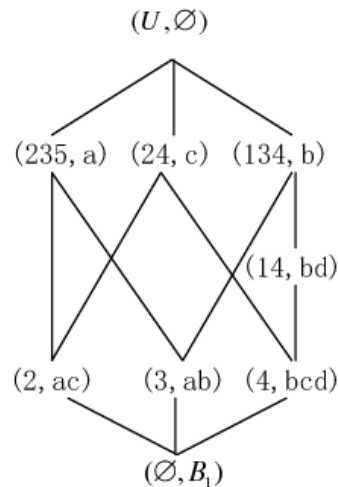
**Example 1** Table 1 gives a formal context  $(U, A, I)$  with  $U = \{1, 2, 3, 4, 5\}$  and  $A = \{a, b, c, d, e\}$ . And the concept lattice of  $(U, A, I)$  is shown in Figure 1.

**Table 1. A Formal Context  $(U, A, I)$**

$U$	$a$	$b$	$c$	$d$	$e$
1	0	1	0	1	0
2	1	0	1	0	1
3	1	1	0	0	1
4	0	1	1	1	0
5	1	0	0	0	1



**Figure 1. Concept lattice  $L(U, A, I)$**



**Figure 2. Concept lattice  $L(U, B_1, I_{B_1})$**

By formula (5), the congruence classes can be computed as follows:

$$\begin{aligned}
 E_1 &:= [1]_{R^A} = \{1, 14\}, & E_2 &:= [2]_{R^A} = \{2\}, & E_3 &:= [3]_{R^A} = \{3\}, \\
 E_4 &:= [4]_{R^A} = \{4\}, & E_5 &:= [5]_{R^A} = \{5, 23, 25, 35, 235\}, \\
 E_6 &:= [13]_{R^A} = \{13, 34, 134\}, & E_7 &:= [24]_{R^A} = \{24\}, \\
 E_8 &:= [12]_{R^A} = \{12, 15, 45, 123, 124, 125, 135, 145, 234, 245, 345, \\
 & \quad 1234, 1235, 1245, 1345, 2345, U\},
 \end{aligned}$$

Table 2 shows the corresponding discernibility matrix  $D$  of  $(U, A, I)$ .

**Table 2. The Discernibility Matrix  $D$  of  $(U, A, I)$**

	$E_1$	$E_2$	$E_3$	$E_4$	$E_5$	$E_6$	$E_7$	$E_8$	$E_9$
$E_1$	$\emptyset$	A	{a,d,e}	{c}	{a,b,d,e}	{d}	{b,c,d}	{b,d}	{a,c,e}
$E_2$		$\emptyset$	{b,c}	{a,b,d,e}	{c}	{a,b,c,e}	{a,e}	{a,c,e}	{b,d}
$E_3$			$\emptyset$	{a,c,d,e}	{b}	{a,e}	{a,b,c,e}	{a,b,e}	{c,d}
$E_4$				$\emptyset$	A	{c,d}	{b,d}	{b,c,d}	{a,e}
$E_5$					$\emptyset$	{a,b,e}	{a,c,e}	{a,e}	{b,c,d}
$E_6$						$\emptyset$	{b,c}	{b}	{a,c,d,e}
$E_7$							$\emptyset$	{c}	{a,b,d,e}
$E_8$								$\emptyset$	A
$E_9$									$\emptyset$

According to Theorem 4 and Table 2,  $B_1 = \{a, b, c, d\}$  and  $B_2 = \{b, c, d, e\}$  are the attribute reducts of  $(U, A, I)$ . By Definition 5,  $a$  and  $e$  are relative necessary attributes;  $b, c$  and  $d$  are core attributes. The concept lattice  $L(U, B_1, I_{B_1})$  is shown in Figure 2. It is easy to see that the two concept lattices are isomorphic to each other.

**Definition 12** Let  $(U, A, I)$  be a formal context. The discernibility function of  $(U, A, I)$  is defined as:

$$f(D) = \bigwedge_{D \in \mathcal{D}, D \neq \emptyset} (\bigvee_{a \in D} a).$$

**Theorem 8** Let  $(U, A, I)$  be a formal context.  $B \subseteq A$  is an attribute reduct of  $(U, A, I)$  iff  $\bigwedge_{a \in B} a$  is a prime implicant of the discernibility function  $f(D)$ .

*Proof.* Necessity. Assume  $B \subseteq A$  is an attribute reduct of  $(U, A, I)$ . By Theorem 4,  $\forall D \in \mathcal{D}, D \neq \emptyset, B \cap D \neq \emptyset$  holds. Then,  $\forall a \in B$ , there exists  $D \in \mathcal{D}, D \neq \emptyset$  such that  $B \cap D = \{a\}$ . Otherwise, if  $|B \cap D| \geq 2$  for all  $D \in \mathcal{D}$  with  $a \in D$ , then  $B - \{a\}$  is also a consistent set of  $(U, A, I)$ , which is a contradiction to that  $B$  is an attribute reduct. It follows that  $\bigwedge_{a \in B} a$  is a prime implicant of  $f(D)$ .

Sufficiency. If  $\bigwedge_{a \in B} a$  is a prime implicant of  $f(D)$ , then  $\forall a \in B$ , there exists  $D \in \mathcal{D}, D \neq \emptyset$  such that  $B \cap D = \{a\}$ . By Theorem 4,  $B - \{a\}$  is not a consistent set of  $(U, A, I)$ . Therefore,  $B$  is an attribute reduct of  $(U, A, I)$ .  $\square$

**Example 2** The discernibility function  $f(D)$  of  $(U, A, I)$  in Example 1 is calculated as follows:

$$\begin{aligned}
 f(D) &= \bigwedge_{D \in D, D \neq \emptyset} (\bigvee_{a \in D} a) \\
 &= (a \vee d \vee e) \wedge (b \vee c) \wedge c \wedge (a \vee b \vee d \vee e) \wedge (a \vee c \vee d \vee e) \wedge b \wedge d \\
 &\quad \wedge (a \vee b \vee c \vee e) \wedge (a \vee e) \wedge (c \vee d) \wedge (a \vee b \vee e) \wedge (b \vee c \vee d) \\
 &\quad \wedge (b \vee d) \wedge (a \vee c \vee e) \wedge (a \vee b \vee c \vee d \vee e) \\
 &= b \wedge c \wedge d \wedge (a \vee e) \\
 &= (a \wedge b \wedge c \wedge d) \vee (b \wedge c \wedge d \wedge e)
 \end{aligned}$$

Therefore, the formal context  $(U, A, I)$  has two reducts. They are  $\{a, b, c, d\}$  and  $\{b, c, d, e\}$  respectively.

## 5. Conclusion

This paper has developed notions and methods of attribute reduction in a formal context based on congruence relations, in order to find the similar expressions with attribute reduction in rough set theory. Discernibility matrices and Boolean functions have been subsequently defined to calculate all attribute reducts. Using the notions of attribute reduction proposed in this paper, methods and results of attribute reduction in a formal context can be derived directly by those in rough set theory. Basing on the reduction in this paper, we can study knowledge reduction in consistent and inconsistent formal decision contexts in further research.

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