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Some Common Aspects of Quantum and Classical Interferences

Youssef Khmou

Sultan Moulay Slimane University, Morocco
khmou.y@gmail.com

Abstract

In quantum information theory, the particles are characterized by wave behavior where they can be described by particle-wave duality, from this perspective, the waves phenomena were studied theoretically and experimentally for quantum particles such as diffraction and interferences. In the other hand, the classical electromagnetic and acoustics fields are described by wave forms where all wave phenomena are possible including diffraction, refraction and scattering. Focusing on interferences where the superposition principle is valid for both quantum and classical interferences, we present in this paper, some common properties between the quantum interferences and particular case of classical interferences. Based on far field and narrowband classical interferences, the interactions between elementary wave forms are treated using array signal processing, where angular interferometry of wave fields using closely sensors or antennas, permit to localize and estimate the characteristics of the far field sources. Based on the geometry of the displacement of the isotropic and identical antennas and the planar shape of the wave front passing near the array, we present some mathematical common concepts between quantum and classical interferences starting from wave propagation equation to the Von Neumann entropy, we present a detailed description of similarities between the two branches using some tools of operator theory. In the last part, we present a numerical simulation where we show that the introduction of the quantum entropy on classical system of antennas-sources permit to identify different phases of the system with respect to the signal to noise ratio.

Keywords: *Quantum interferences, classical interferences, propagation, antennas, sources, state vector, entropy, superposition, operator*

1. Introduction

The branch of science that treats the dynamics of microscopic particles is quantum mechanics [1,2], it is based on operator theory [3] where the operator's characteristics are the outcomes of the measurements of the experience. The wave-particle property [4] is fundamental concept in quantum mechanics where a particle is described by wave function, the link between wave propagation theory and quantum mechanics was given by De Broglie [5] where the momentum of the particle is related to the associated wave vector via Planck's constant. Given this property, many contributions have demonstrated that the square of absolute value of the wave function in classical domain is equivalent to probability density function in quantum mechanics [6], and the concepts of interferences such as temporal coherence, spatial coherence [7] and interaction between several classical wave functions, have meaning on the microscopic level. The most common example for such correspondence between classical and quantum interferences is the double aperture problem [8] that demonstrated the principle of superposition and the meaning of the spatial or momentum wave functions of particles such as photons or electrons.

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In classical interferences, the dynamics of waves are studied using vector formalism based on Maxwell's equations [9] that permit to consider the different states of wave fronts such as multiple interferences and polarization [10] which can be linear, circular, elliptical or random. In quantum mechanics, the classical quantities are replaced by symmetric operators such that the eigenvalues represent the results that we obtain if we make measurements in laboratory.

The purpose of this paper is to present the common mathematical aspects of quantum and classical interferences using operator theory, we present some analogies between the two fields based on a case study of classical angular interferometer system, staring from the equation of propagation to the concept of quantum entropy, we present a common underlying mathematical formalisms.

2. Case study of classical interferences

In this section, we explain the necessary conditions to describe some properties of classical interferences and the parameters we are interested in. By classical interferences, we mean the interaction of classical field, precisely electromagnetic field with measurement devices, where the variation of the field, when it passes through the device, generates an output by which interference properties are exploited to measure some characteristics of the field such as the localization of the wave front. The field is transverse mode where we focus only on the electric field E whose variation is perpendicular to the direction of motion, the field is generated from a source whose dimension is infinitesimal such as punctual source, a simple example is dipole antenna [11] located in position (x_0, y_0) that generates omnidirectional radiation in (x, y) plan, this means that intensity of the field in independent of the azimuth angle [11].

The study of classical interferences of the field is made in Fraunhofer region of propagation, also called far field region [12], using a collection of measurement devices such as dipole antennas placed with respect to some geometric constraints, for simplicity we consider the case of an antenna that consists of N dipoles where the consecutive distance d is uniform and equals half the wavelength λ of the radiation coming from the source, such configuration is called Uniform Linear Array (ULA) [11,12,13], given the length of the antenna as $D=(N-1)d$, the far field region is given by:

$$r_0 \gg \frac{2D^2}{\lambda} \tag{1}$$

Where $r_0=\sqrt{x_0^2+y_0^2}$. It is assumed that the field E is vertically polarized and the system source-antennas are vertically oriented so as to establish a compatibility of measurements. The radiating source can be characterized by many parameters such as the frequency spectrum, the power of radiation related to Poynting vector [14], the temporal coherence, the spatial coherence, the correlation function and the position of the source r_0. For simplicity, we consider only the azimuth angle θ to be our parameter of interest, which is measured with respect to the normal of array of dipoles placed in the origin of Cartesian reference. Given these conditions we begin the illustration of the common properties between quantum theory and classical interferences where we present in the next section, the equation of propagation.

3. Equation of Propagation

The first tool that we need understand is the mechanism of motion which is explained via the propagation equation, the solution of such equation gives a function that can be used to study the dynamics of a wave or a particle. For both fields, the propagation equations are different however the structures of the solutions are the same. Let us begin with quantum mechanics, for simplicity we consider the one dimensional problem, the particle such as electron is described by diffusion equation known as Schrödinger's equation [15]:

$$i\hbar\frac{\partial\Psi}{\partial t}=\frac{-\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}+V(x)\Psi \tag{2}$$

Where $i^2=-1$, $\hbar$ is the reduced Planck's constant and $V(x)$ is the potential. To illustrate the significance of the potential, we consider the case of diffraction with an aperture of width a, the potential in this case is $V\left(x\right)=0$ if $0\le x\le a$ and $V(x)=+\infty$ for $x<0$ and $x>a$ which represents the opaque surface where a photon cannot pass through. The solution of the above equation can be written in the form:

$$\Psi=\Psi_0 e^{j((E/\hbar)t-(p/\hbar)x)} \tag{3}$$

Where Ψ_0 is the constant amplitude, E is the associated energy to the particle, for photons the energy is $E=h\nu$ where ν is the central frequency with small bandwidth. p is the momentum, x the direction of the propagation and t is the time index. This solution is also the same for classical field, considering the electric part of electromagnetic wave that is vertically polarized, the derivation of the propagation equation [12] is of second order temporally and spatially:

$$\frac{\partial^2 E}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} \tag{4}$$

Where c the phase speed relatively to the medium of propagation, for vacuum it is related to permittivity ε_0 [9] and permeability μ_0 [9] by the relation [16]:

$$c=\frac{1}{\sqrt{\varepsilon_0\mu_0}} \tag{5}$$

The solution of the equation is also written as the following:

$$E=E_0 e^{j(2\pi\nu t-kx)} \tag{6}$$

Where E_0 is the amplitude, ν is the central frequency of the field, and k is the wave number, in two dimensional plan, the wave vector is written as:

$$\vec{k}=\frac{2\pi}{\lambda}(\cos(\theta)\vec{e}_x+\sin(\theta)\vec{e}_y) \tag{7}$$

Where λ is the wavelength, and $\left\|\vec{k}\right\|=2\pi\lambda^{-1}$. For both solutions, we made the approximation related to theirs amplitude where we considered that E_0 and Ψ_0 are constant, which represents the ideal monochromatic fields, in reality the mechanism of radiation either on microscopic level (atom for example) or macroscopic level (antenna for example) is with finite duration, which means that the frequency spectrum contains a bandwidth, therefore E_0 and Ψ_0 depend on x and t with a slow variation comparatively to the frequency ν, this approximation is also known as Slowly Varying Envelope Approximation [17]. In terms of temporal dimension, the lower value for the uncertainty of the frequency and time standard deviations denoted by $\Delta\omega$ and Δt respectively is given by the relation [18]:

$$\Delta\omega\Delta t \geq \frac{1}{2} \tag{8}$$

The link between the two wave functions is made via De Broglie relation [5] that associates a characteristic of wave to the particle, from mathematical viewpoint, the relations of the passage from quantum to classical expression are listed as follows:

$$\begin{cases} \Psi_0 \to E_0 \\ p = \hbar k \\ \dfrac{E}{\hbar} = 2\pi\nu \end{cases} \tag{9}$$

The spatial version of the uncertainty principle using wave number and position standard deviations is $\Delta k\Delta x \geq (1/2)$ which can be derived using one of the Heisenberg's uncertainty principles [19,20] $\Delta p\Delta x \geq (\hbar/2)$ or $\Delta E\Delta t \geq (\hbar/2)$ for energy and time variables. Generally, for bandwidth $\Delta\nu$ the approximation of the constant envelope can be made if the following condition [21] is verified:

$$\frac{\Delta\nu}{\nu} \ll 1 \tag{10}$$

Where $\Delta\nu = \nu_{max} - \nu_{min}$. The solution of both equations permit to study the properties of wave and particles, for example the electron diffraction [22] can be described by the wave functions $\Psi(x)$ or $\Psi(p)$ that are related by Fourier transform:

$$\Psi(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty}\Psi(x)e^{-ipx/\hbar}dx \tag{11}$$

Where the distribution of the spatial wave function indicates the repartitions of the particles on screen of observation, such that the square of the absolute value of $\Psi(x)$ represents the probability density function with normalization condition:

$$\int_{-\infty}^{+\infty}\Psi(x)\Psi^*(x)dx = 1 \tag{12}$$

Where $(.)^*$ is the conjugate operator. The equivalent of the square of the absolute value of $\Psi(x)$, in the classical domain is the intensity, which illustrates the physical interpretation of spatial coherence via visibility for visible light waves, the intensity [23] observed on screen in coordinate x is given by:

$$I(x) = \frac{1}{2}\varepsilon_0 c < EE^* >= \frac{\varepsilon_0 c}{2T}\int_{t-T/2}^{t+T/2} EE^* dt \tag{13}$$

Where T is the observation period, note that this definition is also valid for other range of frequencies. As we are interested in Radio frequency range, we study only some parameters related to angular interferometer systems [24] of radio waves where we present some common mathematical formalism between classical and quantum theories. In the next section, we present the notion of the state vector.

4. State Vector

In quantum information theory [25], the state vector given in N dimensional Hilbert space $|\Psi> \in C^{N\times 1}$ contains the necessary information about the studied system where different components α_i represent the coefficients of the different states of the system:

$$|\Psi> = \sum_{i=1}^{N} \alpha_i \, |u_i> \tag{14}$$

Where the vectors $\{|u_i>\}$ form an orthonormal base that is described by the following property:

$$< u_i \, | \, u_j > = \sum_{k=1}^{N} u_{k,i}^{*} u_{k,j} = \delta_{ij} \tag{15}$$

Such that δ_{ij} is the Kronecker symbol $\delta_{ij} = 1$ if $i = j$ and 0 otherwise, the state vector is normalized via the condition:

$$\sum_{i=1}^{N} |\alpha_i|^2 = 1 \tag{16}$$

This notion of state vector is also valid for particular case of classical interferences, given a group of antennas clustered as an uniform linear array [11,12,13] where the antennas are identical and isotropic, a planar wave field passing through the array induces a voltage at each antenna with progressive phase, for better illustration we present in Figure 1 different orientations of the planar wave field coming from far field punctual source.

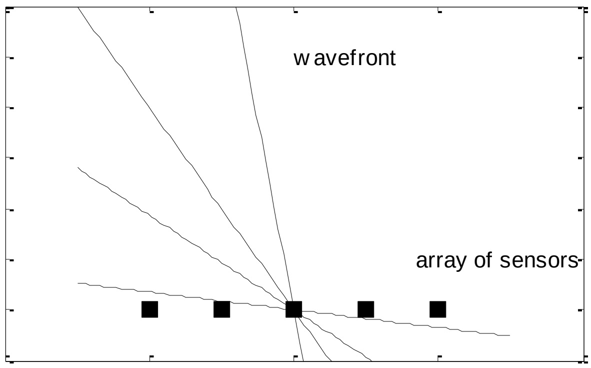

Figure 1. An Array of Sensors and Four Different Orientations of Planar Wave Front

For simplicity, we consider that the signal at the first sensor is denoted by $s_1(t)$. Using narrowband condition [11,21], the signal at second sensor is that of the first sensor with phase delay $s_2(t) = s_1(t)e^{-j\phi}$. Using simple geometrical relation, the phase difference is related to the angle of incidence of the wave field as $\phi = kd\sin(\theta)$ where d is the uniform distance between sensors and θ is the angle of incidence of the field relatively to the normal axis of the array, following the same expression for the different sensors

$s_i(t)$, we obtain a vector of the phase delay $a \in C^{N\times 1}$ that depends on the geometry of the array [12], it is written as:

$$a(\theta) = \begin{pmatrix} 1 \\ e^{-jkd\sin(\theta)} \\ \vdots \\ e^{-jkd(N-1)\sin(\theta)} \end{pmatrix} \tag{17}$$

$a(\theta)$ is equivalent of the state vector in quantum information theory, in this case, the knowledge of the angle of incidence is sufficient to construct $a(\theta)$. Indeed, given a superposition of P wave fields, the generated signals from the array with K samples is complex matrix $X(t) \in C^{N\times K}$ where the m^{th} element is written as [11]:

$$x_m(t) = \sum_{g=1}^{P} s_g(t) e^{-jkd(m-1)\sin(\theta_g)} + n_m(t) \tag{18}$$

Where the system is defined by the set $[\theta_1, ..., \theta_P]$, $n_m(t)$ is a model of the additive perturbing noise [11] generated by internal factors such as thermal noise and external factors such as interferences from secondary sources. $n_m(t)$ is described by complex zero mean random process. In matrix form the signals are written as:

$$X(t) = AS(t) + n(t) \tag{19}$$

Where $A \in C^{N\times P}$ is called steering matrix [12] of the array, $S(t) \in C^{P\times K}$ is the matrix of the envelopes of sources and $n(t) \in C^{N\times K}$ is complex random matrix with no correlation between rows and columns. The steering matrix is given as $A = [a_1, ..., a_P]$ where each column represents a steering vector that describes the angular state of the incident wave front:

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ e^{-jkd\sin(\theta_1)} & e^{-jkd\sin(\theta_2)} & \cdots & e^{-jkd\sin(\theta_P)} \\ \vdots & \vdots & \vdots & \vdots \\ e^{-jkd(N-1)\sin(\theta_1)} & e^{-jkd(N-1)\sin(\theta_2)} & \cdots & e^{-jkd(N-1)\sin(\theta_P)} \end{pmatrix} \tag{20}$$

The radiating source is classified by its statistical and angular parameters such as azimuth and elevation angles, using the data generated by the array of antennas, the importance of the state vector $a(\theta)$ resides in the angular scan where the peak of the spectrum indicates an estimate of the angle of incidence of the field. The second order statistic of data $X(t)$ [11,12] is given by the relation:

$$\Gamma = < XX^{+} > = A < SS^{+} > A^{+} + \sigma^2 I_N \tag{21}$$

Where $< SS^{+} >$ is the correlation between the envelopes of the sources, σ^2 is the noise power and I_N is the identity matrix. Given the case where the P waveforms sources are not correlated but may have different levels of power, the correlation matrix is written as:

$$< SS^+ >= \begin{pmatrix} \sigma_1^2 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_P^2 \end{pmatrix} \quad (22)$$

Digital processing of data (X,Γ) with angular region $\Omega=[\theta_{min},\theta_{max}]$ gives a spectrum $f(\theta)$ where using as example the beam forming method, for each value of angle $\theta \in \Omega$, an electronic steering vector $a(\theta)$ is generated and the spectrum is computed by the relation:

$$f(\theta) = \frac{1}{N^2} a^+(\theta)\Gamma a(\theta) \quad (23)$$

Thus, whenever the tested angle if the true angle of incidence of the field, $f(\theta)$ has peak which enables the identification of the angular parameters of the sources. Note that the spectrum given above is of low resolution, which means that two sources with angular difference $|\theta_i - \theta_j|$ less than Rayleigh angular resolution limit of the array [14] which is approximately given by $\theta_R = \lambda / (N-1)d$, the spectrum cannot separate the peaks, therefore other methods called high resolution techniques [11,12] enable this process. Based on this comparison, we remark the importance of the state vector in quantum and classical fields where the vector contains necessary information of the parameters of the system, where in classical field, the system consists of a configuration of antennas-sources. Given the second order statistic of data, we present in the next section some common properties between the covariance matrix in terms of eigenvalues and the Hamiltonian operator in quantum mechanics.

5. Hermitian Operator

The harmonic oscillator model is used in quantum mechanics to obtain the solution of the eigenvalue problems, in one dimensional system, the Hamiltonian is given [23] by:

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \quad (24)$$

m is the mass of the particle and $\omega = 2\pi\nu$ is the angular frequency, x is the position and $p = -i\hbar\partial / \partial x$ is the momentum where the two variables verify the relation:

$$[x, p] = xp - px = i\hbar \quad (25)$$

The quantization of the harmonic oscillator consists of transforming the variables of the Hamiltonian into operators, let us consider the ladder-operators:

$$\begin{cases} a = \sqrt{\dfrac{m\omega}{2\hbar}} \left(x + \dfrac{ip}{m\omega} \right) \\ a^+ = \sqrt{\dfrac{m\omega}{2\hbar}} \left(x - \dfrac{ip}{m\omega} \right) \end{cases} \quad (26)$$

a is called annihilation operator and its transpose-conjugate a^+ is called creation operator. Using the expression of the momentum p, the operators verify the following commutation relation:

$$[a, a^+] = 1 \qquad (27)$$

The Hamiltonian can be rewritten using the creation and annihilation operators as:

$$H = \hbar\omega\left(a^+ a + \frac{1}{2}\right) \qquad (28)$$

The eigenvalue problem is described by the equation $H \mid u_n >= E_n \mid u_n >$ with eigenvector $\mid u_n >$ and eigenvalue E_n given in general form by:

$$E_n = (n + \frac{1}{2})\hbar\omega \qquad (29)$$

The minimum value of the energy level is $E_0 = (1/2)\hbar\omega$, and all the eigenvalues are real and positive such that H is hermitian operator $H^+ = H$ .Similarly, the covariance matrix is hermitian operator $\Gamma^+ = \Gamma$ [9,13] with positive and real eigenvalues. The mean value of the operator H in state $\mid \Psi >$ given by $< \Psi \mid H \mid \Psi >$ is equivalent to the spectrum of the beamforming [11,12] in antenna processing given in equation (23) by $< a \mid \Gamma \mid a >$ with proportional constant $(1/N^2)$. Given P sources, in the absence of noise $n(t)$, the spectrum of the covariance matrix is given by [11]:

$$\sigma_\Gamma = [\lambda_1, ..., \lambda_P, 0_{1\times N-P}] \qquad (30)$$

The minimum eigenvalue is degenerate $N-P$ times, in the presence of noise, the general structure of the spectrum is given by [12]:

$$\sigma_\Gamma = [\lambda_1 \geq \ ... \geq \lambda_P > \lambda_{P+1} \square ... \square \lambda_N = \sigma^2] \qquad (31)$$

The spectral decomposition of the covariance matrix can be divided into two subspaces called noise and signal subspaces, the first set corresponds to the first P largest eigenvalues and the noise subspace corresponds to the smallest eigenvalues $[\lambda_{P+1}, ..., \lambda_N]$, the spectral decomposition of the covariance matrix is given by [11]:

$$\Gamma = \sum_{i=1}^{N} \lambda_i \mid u_i >< u_i \mid \qquad (32)$$

The signal subspace is formed by the set of the first P eigenvectors:

$$U_s = [\mid u_1 >, ..., \mid u_P >] \qquad (33)$$

And the noise subspace is formed by the set of the $N-P$ remaining eigenvectors:

$$U_n = [\mid u_{P+1} >, ..., \mid u_N >] \qquad (34)$$

The concept of projectors in quantum mechanics can be applied in this case of classical interferences by defining the projectors into the noise and signal subspaces using the following projector operators:

$$\begin{cases} P_n = U_n U_n^+ \\ P_s = U_s U_s^+ \end{cases} \tag{35}$$

Based on the relation $\sum_{i=1}^{N} |u_i >< u_i| = I_N$ where the eigenvectors are orthonormal and form the matrix U which is unitary [11] $U^+ = U^{-1}$, we derive the following properties of the two projectors:

$$\begin{cases} P_s P_n = 0_{N\times N} \\ P_s + P_n = I_N \\ (P_n + P_s)^m = \sum_{k=0}^{m} C_m^k P_n^k P_s^{m-k} = P_s^m + \sum_{k=1}^{m-1} C_m^k P_n^k P_s^{m-k} + P_n^m = I_N \\ Tr(P_s) = P \\ \|P_s\|_F = \sqrt{P} \\ Tr(P_n) = N - P \\ \|P_n\|_F = \sqrt{N-P} \end{cases} \tag{36}$$

The operator $\|.\|_F$ denotes the Frobenius norm, for steering matrix $A \in C^{N\times P}$, the Frobenius norm is calculated by the equation:

$$\|A\|_F = \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{P} |A_{ij}|^2} = \sqrt{Tr(A^+ A)} = \sqrt{NP} \tag{37}$$

The orthogonal property between the sets of signal and noise subspaces allows to identify the angles of incidence [11,12,13] of the wave front and extract other properties of sources such as the powers of radiation and the shapes of the waveforms, the orthogonal property is given as [11]:

$$\{| a(\theta_1) >, ..., | a(\theta_P) >\} \perp \{| u_{P+1} >, ..., | u_N >\} \tag{38}$$

To put the ensemble of discussed similarities and concepts in summarized form, we present in table.1 the different components of quantum information system and their corresponding equivalent concepts of classical interferences.

Table 1. Different Concepts of Particular Case in Quantum Interferences and their Corresponding Parameters in Classical Domain

Variable	**Quantum interferences**	**Classical interferences**
States	Number of states N	Number of sensors N
Dynamical Equation	$i\hbar \frac{\partial \Psi}{\partial t} = \frac{-\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V(x)\Psi$	$\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}$
Waveform	$\Psi = \Psi_0 e^{j((E/\hbar)t-(p/\hbar)x)}$	$E = E_0 e^{j(2\pi\nu t - kx)}$
Wave number	$k = \frac{p}{\hbar}$	$k = \frac{2\pi}{\lambda}$
Uncertainty principle I	$\Delta p \Delta x \geq \frac{\hbar}{2}$	$\Delta k \Delta x \geq \frac{1}{2}$

Uncertainty principle II	$\Delta E \Delta t \geq \frac{\hbar}{2}$	$\Delta \omega \Delta t \geq \frac{1}{2}$
Linear superposition	$\Psi = \sum \Psi_i$	$E = \sum E_i$
Principal operator	$H = T + V$	$\Gamma = A < SS^+ > A^+ + \sigma^2 I_N$
Hermitian operator	$H^+ = H$	$\Gamma^+ = \Gamma$
Eigenvalue equation	$H \| u_n >= E_n \| u_n >$	$\Gamma \| u_i >= \lambda_i \| u_i >$
Minimum eigenvalue	$E_0 = (1/2)\hbar\omega$	$\lambda_{min} = \sigma^2$
Orthonormal base	$\sum_{n=1}^{N} \|u_n >< u_n \| = I_N$	$\sum_{i=1}^{N} \|u_i >< u_i \| = I_N$
State vector	$\| \Psi >$	$\| a(\theta) >$
Vector norm	$\\| \| \Psi > \\|_F = 1$	$\\| \| a(\theta) > \\|_F = \sqrt{N}$
Expectation value	$< \Psi \| H \| \Psi >$	$< a(\theta) \| \Gamma \| a(\theta) >$
Density operator	$\rho = \sum_{i=1}^{N} p_i \| u_i >< u_i \|$	$\rho = \frac{\Gamma}{Tr(\Gamma)}$

Remark: In the expression of the Hamiltonian, the operators T and V refer to the kinetic and potential energies respectively and the expression of the covariance matrix is combination of signal operator $A < SS^+ > A^+$ and noise operator $\sigma^2 I_N$. The density operator given in the last line will be explicated in the next section.

Let us now identity the possible configurations of the system composed of antennas and sources, with respect to the number of the sources and the absence and presence of noise. By excluding the cases of correlation between different sources [11] $< s_i(t) s_j^*(t) >$, the spatial correlation between noise waveforms for N channels $< n_i(t) n_j^*(t) >$ [11,13] and the correlation between the sources and noise $< s_i(t) n_j^*(t) >$ [13], we consider the following properties :

$$\begin{cases} < n_i(t) n_i^*(t+\tau) >= \sigma^2 \delta(\tau) \\ < n_i(t) n_j^*(t) >= \sigma^2 \delta_{ij} \\ < s_i(t) n_j^*(t) >= 0 \\ < n(t) n^+(t) >= \sigma^2 I_N \end{cases} \quad (39)$$

we can identify the following cases where the above three statistical measures are uncorrelated:

- N antennas and P sources where $1 < P < N$ in the presence of noise, this is the usual case where the performances of the high resolution techniques [11] are evaluated with respect to the noise level that can be expressed in terms of signal to noise ratio defined by $SNR_i = 10 \log_{10}(\sigma_i^2 / \sigma^2)$ for the i^{th} source.
- N antennas and P sources where $1 < P < N$ and the absence noise $n(t) = 0_{N \times K}$, this case can be realized in the laboratory using absorbing boundaries where the interferences are absent, this case corresponds to $SNR \to +\infty$. However, in real

experiments the perturbing noise is always present even in the absence of primary radiating sources.

- N antennas and single source $P=1$, in the presence of noise, the spectrum of the covariance matrix contains one largest eigenvalue and the $N-1$ remaining eigenvalues correspond to the noise power σ^2.
- N antennas and single source P=1, in the absence of noise $n(t)=0_{N\times K}$, this case can be realized in anechoic chamber where the single non zero eigenvalue equals $N\sigma_1^2$ as we will see in the next section.
- N antennas and absence of sources $P=0$, this case is useful to make measurements of the ambient noise and estimate the noise power σ^2, this parameter can be useful for certain high resolution techniques for angle of arrival estimation where the angular spectrum is based on an operator constructed using a threshold λ_c between signal and noise eigenvalues, this operator is given by the equation:

$$H=\lim_{m\to+\infty}\left(\left(\frac{\Gamma}{\lambda_c}\right)^m+I_N\right)^{-1} \tag{40}$$

where the threshold parameter of the eigenvalues λ_c is in the range:

$$\lambda_c \in]\lambda_{P+1},\lambda_P[\tag{41}$$

Note that the case where the number of radiating sources is higher than the number of antennas $P\geq N$ is not discussed in this paper as well as the case of $(N,P=0,\sigma^2=0)$. As summary of the mentioned cases, we present in Table 2 the configurations and the corresponding values of the spectrum of Γ.

Table 2. Different Configurations of antennas-sources (N,P,σ^2) and their Corresponding Spectra of the Covariance Matrix $\Gamma=\langle XX^+\rangle$

Configuration of antennas-sources	Spectrum of covariance matrix Γ
$(N,1<P<N,\sigma^2)$	$[\lambda_1\geq\ ...\geq\lambda_P>\lambda_{P+1}\square ...\square\ \lambda_N=\sigma^2]$
$(N,1<P<N,\sigma^2=0)$	$[\lambda_1,...,\lambda_P,0_{1\times N-P}]$
$(N,P=1,\sigma^2)$	$[\lambda_1>\lambda_2\square ...\square\ \lambda_N=\sigma^2]$
$(N,P=1,\sigma^2=0)$	$[\lambda_1>\lambda_2=...=\lambda_N=0]$
$(N,P=0,\sigma^2)$	$[\lambda_1\square\ \lambda_2\square ...\square\ \lambda_N=\sigma^2]$

Given the different possible configurations, we remark that each case can be characterized in terms of the spectrum of the covariance matrix, similarly to the notion of mixed and pure states [23] in quantum information theory, we can implement some quantum tools to identify the pure and mixed states in classical interferences, precisely the quantum entropy which will be the subject of the next section.

6. Von Neumann Entropy

In quantum information theory, the state of the system is described by state vector $|\Psi_i>$ say for example the polarization state of photons, when the system is in mixed state, the ensemble of N states can be represented using the density operator [23] where each state is given with probability p_i as:

$$\rho = \sum_{i=1}^{N} p_i \,|\,\Psi_i >< \Psi_i\,| \tag{42}$$

With normalization condition $\sum_{i=1}^{N} p_i = 1$. One way to characterize the system is by the Von Neumann entropy which is a generalization of the classical entropy in information theory, it is given by the equation:

$$h(\rho) = -\sum_{i=1}^{N} p_i ln(p_i) = -Tr(\rho ln(\rho)) \tag{43}$$

If the system is in pure state then the entropy equals zero where the density operator verifies $\rho^n = \rho$. In mixed state case, $h(\rho)$ is always positive and reaches maximal value $ln(N)$ if the system is in totally mixed state. An example of the operator that corresponds to the maximal value of the entropy is $\rho = (1/N)I_N$. Among the properties of the quantum entropy is the invariance under unitary transformation, for example given a unitary operator $Q \in C^{N\times N}$ where it verifies the property $Q^+ = Q^{-1}$, the entropy is invariant $h(Q\rho Q^+) = h(\rho)$. Because $h(\rho)$ depends only on the eigenvalues, further calculation shows that:

$$Q\rho Q^+ = Q\sum_{i=1}^{N} p_i \,|\,u_i >< u_i\,|\,Q^+ = \sum_{i=1}^{N} p_i \,|\,v_i >< v_i\,| \tag{44}$$

$\{|\,v_i >= Q\,|\,u_i >\}$ is an orthonormal basis $\sum_{i=1}^{N} |v_i >< v_i| = Q\sum_{i=1}^{N} |u_i >< u_i\,|\,Q^+ = I_N$. Note that the entropy is calculated using the convention $0\ ln(0) = 0$.The entropy function can indeed be used to identify the pure state in classical interferences, precisely a configuration of antennas-sources where the Von Neumann entropy is zero. From the previously cited configurations, we already remark that the case of single source in the absence of noise $(N, P = 1, \sigma^2 = 0)$ can be considered as pure state, let us first explicit the structure of the covariance matrix, given single source the steering matrix becomes vector$|\,a(\theta)>$, with angle of incidence θ, that verifies the property:

$$< a(\theta)\,|\,a(\theta) >= \sum_{g=1}^{N} e^{-jkd(g-1)\sin(\theta)} e^{jkd(g-1)\sin(\theta)} = N \tag{45}$$

Where the envelope of the sources has power $< s_1 s_1^* >= \sigma_1^2$.The covariance matrix in the absence of noise $n(t) = 0_{N\times K}$ is calculated as:

$$\Gamma = \sigma_1^2\,|\,a(\theta) >< a(\theta)\,| \tag{46}$$

Which is dyadic operator, in fact an operator X that equals a product $|x><x|$ has one non zero eigenvalue whose magnitude is $\lambda_1 =<x|x>$, therefore the single non zero eigenvalue of the covariance matrix is $\lambda_1 = N\sigma_1^2$ where the spectrum as mentioned in table.2 is given by the vector $[N\sigma_1^2, 0_{1\times N-1}]$. To proceed with quantum entropy, we must first transform the covariance matrix into density operator by the relation:

$$\rho = \frac{\Gamma}{Tr(\Gamma)} = \frac{\Gamma}{N\sigma_1^2} \tag{47}$$

The spectrum of the density operator is $[1, 0_{1\times N-1}]$, and consequently the Von Neumann entropy for this configuration of single source is zero:

$$h(\rho) = -\sum_{i=1}^{N} \lambda_i ln(\lambda_i) = 0 \tag{48}$$

Therefore pure state in classical interferences is a configuration $(N, P=1, \sigma^2 = 0)$ whose quantum entropy is zero similarly to the quantum systems. Other than this particular case, the Von Neumann entropy can be used to identify the phases of the of the system antennas-sources with respect to the variations of the signal to noise ratio. The variation of the noise level implies a change in the order of the eigenvalues of Γ and scalar function such as the entropy can give a global description of the state of the system. To clarify this concept, we present a numerical simulation of particular case of antennas-sources to put into evidence the variation of the quantum entropy.

Let us consider a uniform linear array of $N = 15$ identical and isotropic sensors in plan (x, y), the signals generated by the sensors consist of radiations coming of three far field punctual sources such as dipoles antennas in the same plan (x, y) and operating with wavelength λ, the distance between the sensors is half the wavelength $d = \lambda / 2$. The angles of incidence of the wave front relatively to the normal of the array are $[15^\circ, 40^\circ, 57^\circ]$. The waveforms of the sources have the same power $\sigma_1^2 = 1$W where the envelopes are modeled by zero mean complex random processes $S(t) \sim CN(0, I_3)$. The data matrix is generated using $K = 200$ samples $X \in C^{15\times 200}$. By varying the noise power σ^2 and consequently the signal to noise ration, we compute for each value of SNR the Von Neumann entropy of the density operator $h(\rho) = h(\Gamma / Tr(\Gamma))$, the variation of the entropy is presented in Figure 2.

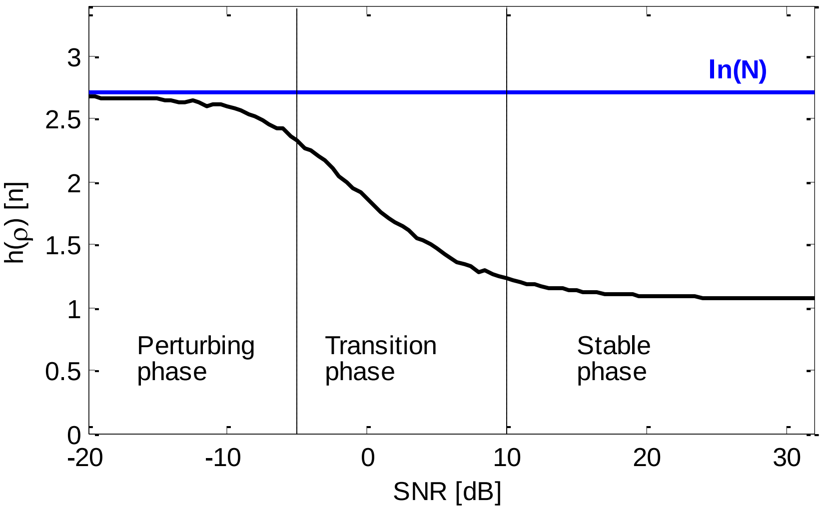

Figure 2. Von Neumann Entropy of Density-Covariance Matrix ρ using a Uniform Linear Array with Configuration $(N=15, P=3, \sigma^2)$

The entropy varies with respect to the noise level, we remark that the variation can be divided into three phases as follows:

P Phase: the perturbing phase that covers the range [-20 dB, -5 dB], it is characterized by higher power of noise than that of sources. In this phase, it is difficult the separate the signal eigenvalues from the noise eigenvalues which implies that it is also difficult to separate the signal and noise subspaces. The differences between the eigenvalues are small which means that entropy $h(\rho)$ tends to is maximal value $ln(N)$ that equals 2.7081 [n] in this case. This remark is in harmony with the principle of mixed states in quantum systems.

T Phase: the transition phase covers approximately the range [-5 dB, 10 dB] , the entropy $h(\rho)$ decreases considerably. In this phase, the resolution of the angles of incidence either by angular spectra or by algebraic techniques depends on the robustness of the employed technique, some methods may offer good resolution than others.

S Phase: the stable phase starts at approximately $SNR = 10$ dB, the function $h(\rho)$ tends to its final value that is approximately 1.1 [n] because of the existence of several sources. Indeed as mentioned in table.2 for configuration $(N, P=1, \sigma^2)$ where the spectrum of the covariance matrix is $[\lambda_1 > \lambda_2 \geq \ldots \geq \lambda_N = \sigma^2]$, as the SNR increases the $N-1$ eigenvalues tend to zero and the spectrum of the density operator $\rho = \Gamma / Tr(\Gamma)$ tends to the distribution $[1, 0 \times_{1 \times N-1}]$, as consequence, the entropy tends to zero, a simple implementation of this case is presented in figure.3 where instead of three sources we keep one source with angle of incidence $\theta = 40^{\circ}$. The variation of the entropy is still described by P, T and S phases as the previous example.

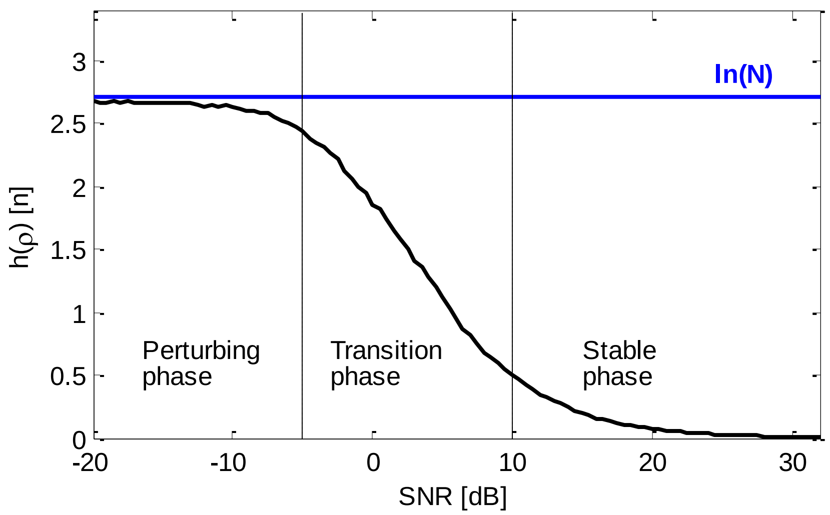

Figure 3. Von Neumann Entropy of Density-Covariance Matrix ρ using a Uniform Linear Array with Configuration $(N=15, P=1, \sigma^2)$

We remark in this second case that the entropy tends to zero starting from 10 dB which means that the configuration tends to the pure state, however it is easy to estimate the characteristics of the single radiating source using only the beamforming spectrum.

7. Conclusion

We have presented, in this paper, some similarities between two particular cases of quantum and classical interferences, using the formalism of array signal processing, we have compared analogous concepts between quantum information system and far field narrowband angular interferometer system in classical fields. Several common properties were explicated including the solutions to the equations of propagation, the state vectors, the Hermitian operators and the eigenvalues equations. The measurements in classical interferences were based on orthogonal properties between the signal and noise subspaces which is concept widely used in quantum field. In the last part, we have applied the quantum entropy on the covariance matrix of the multi channels of the array of sensors where it was shown that the variation of the entropy function can identify three phases of the system antennas-sources relatively to the spectrum of the covariance matrix, with respect to the signal to noise ratio.

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