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Theory of Fuzzy Sets: An Axiomatic Approach

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Abstract

How exactly the membership function of a normal fuzzy number should be determined mathematically was not explained by the originator of the theory. Further, the definition of the complement of a fuzzy set led to the conclusion that fuzzy sets do not form a field. In this article, we would put forward an axiomatic definition of fuzziness such that fuzzy sets can be seen to follow classical measure theoretic and field theoretic formalisms.

Keywords: *Fuzzy membership function, probability measure, complement of a fuzzy set, MSC (2010) 03E72*

1. Introduction

Half a century ago, the theory of fuzzy sets came into existence with the declaration (Zadeh [1], page – 338): 'More often than not, the classes of objects encountered in the real physical world do not have precisely defined criteria of membership.' If it is presumed the there cannot exist a precisely defined criteria of membership, how exactly the membership function would have to be obtained was anyway out of question.

A few years thereafter, Zadeh [2] put forward the concept of possibility distribution associated with a fuzzy number, which was defined as numerically equal to the membership function of the fuzzy number. He forwarded a Possibility-Probability Consistency Principle, according to which a probability space could be imposed on the interval on which a law of fuzziness was defined. How the concerned probability law has to be defined was however not explained. In fact, Zadeh clearly mentioned that the approach towards linking fuzziness with probability was heuristic (Zadeh [2], page – 8): 'It should be understood, of course, that the possibility-probability consistency principle is not a precise law or a relationship that is intrinsic in the concepts of possibility and probability.' Basically, an attempt was made to link fuzziness with the probability measure; the attempt ended in failure.

Regarding the membership function of a fuzzy number, it was however mentioned (Zadeh [1], page – 339): 'Essentially, such a framework provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership rather than the presence of random variables.' However, he put forward a principle linking a law of fuzziness with a law of probability defined on the same interval (Zadeh [2]). If that kind of a link between fuzziness with probability would have been found, that would have given us an idea regarding how to construct the membership functions of a fuzzy sets. In that case, an axiomatic definition of fuzziness would perhaps have been available years ago.

Secondly, the complement A^C of a fuzzy set A was defined as follows set (Zadeh [1], Eq. 1, page – 340): If $f_A(x)$ is the membership function of a normal fuzzy set A, then for A^C the membership function set would be given by

$$f_{A^c}(x) = 1 - f_A(x).$$

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This definition of the complement of a fuzzy set led to the conclusion that the neither the union of a fuzzy set and its complement is the universal set, nor their intersection is the null set. As a result, it was accepted that fuzzy sets do not form a field.

Indeed, partial presence of an element in a set is the basic theme that defines fuzziness. If we can express the level of partial presence mathematically, that would lead to a precise definition of fuzziness. After all, the definition of anything mathematical, including the mathematics of fuzziness, should not be fuzzy. An axiomatic definition of fuzziness is not available in the literature. Just as probability is explained axiomatically, fuzziness too should have an axiomatic definition. In what follows, we shall put forward an axiomatic definition of fuzziness.

2. The Proposed Axiomatic Approach to Define Fuzziness

We would first like to state the original definition of fuzzy sets proposed by the originator of the theory (Zadeh [1], page - 339): 'Let X be a space of points, with a generic element of X denoted by x. Thus $X = \{x\}$. A fuzzy set A in X is characterized by a membership function $f_A(x)$ which associates with each point in X a real number in the interval [0, 1], with the values of $f_A(x)$ at x representing the *grade of membership* of x in A.' It was of course mentioned (Zadeh [1], *e.g.*, page – 339, footnote no. 3) that the membership function is continuous.

Indeed, a fuzzy real number $[\alpha, \beta, \gamma]$ is an interval around the real number β with the elements in the interval being partially present. The level of presence of an element in the interval is reflected in the *membership function*. Let a *normal* fuzzy number $N = [\alpha, \beta, \gamma]$ be associated with the membership function $\mu_N(x)$, where

$$\begin{aligned} \mu_N(x) &= \Psi_1(x), \text{ if } \alpha \le x \le \beta, \\ &= \Psi_2(x), \text{ if } \beta \le x \le \gamma, \text{ and} \\ &= 0, \text{ otherwise.} \end{aligned}$$

According to the Dubois-Prade definition of normal fuzziness, $\Psi_1(x)$ is continuous and non-decreasing in the interval $[\alpha, \beta]$, and $\Psi_2(x)$ is continuous and non-increasing in the interval $[\beta, \gamma]$, with $\Psi_1(\alpha) = \Psi_2(\gamma) = 0$, $\Psi_1(\beta) = \Psi_2(\beta) = 1$. In this definition, it was declared that the membership function should in fact be expressed as two different functions, the Left Reference Function $\Psi_1(x)$ in the interval $[\alpha, \beta]$ and the Right Reference Function $\Psi_2(x)$ in the interval $[\beta, \gamma]$. However, how exactly should we obtain the two functions mathematically was never discussed.

Keeping in view the original definition forwarded by Zadeh regarding continuity of the membership function, and the Dubois-Prade definition of the membership function as two reference functions in two intervals, we would like to put forward the following axiom to define the membership function associated with normal fuzziness.

Axiom - 1: For a normal fuzzy number $N = [\alpha, \beta, \gamma]$, the Dubois – Prade left reference function $\Psi_1(x)$ is the distribution function of a random variable defined in the interval $[\alpha, \beta]$, and the Dubois – Prade right reference function $\Psi_2(x)$ is the complementary distribution function of another random variable defined in the interval $[\beta, \gamma]$.

In fact, the notion of probability does not enter into the definition of a random variable (Rohatgi and Saleh [4], page – 43). This means, measure theoretically, a variable associated with a probability law is necessarily random, but a random variable is not necessarily probabilistic. Therefore a probability density function is necessarily a density function, but a density function defined with reference to a random variable is not necessarily a probability density function. Similarly, a distribution function is not necessarily a probability distribution function, but a probability distribution function is necessarily a distribution function. In Axiom – 1, we have used the terms *random variable* and *distribution function* in the measure theoretic sense.

We would now like to mention one single point about the membership function of the complement of a normal fuzzy set. Not everything can be measured from the zero level. If a person is 6 feet tall, and if he stands on a table 3 feet in height, his height would not be 9 feet, because we would then measure his height not from the ground but from the table. The membership values of the elements of the complement of a normal fuzzy number *A*, in the same way, must be calculated from the respective membership values of *A*. But for the complement of a normal fuzzy set, *membership value* and *membership function* are two different things. Here is where we are disagreeing with Zadeh and others who followed his definition of the membership function of the complement of a normal fuzzy set. Indeed,

$$f_A{}^c(x) = 1 - f_A(x)$$

gives us membership values for any given *x* of the complement of the fuzzy set; it is not the membership function. We are now going to put forward the second axiom towards defining fuzziness.

Axiom - 2: The membership function of the complement of a fuzzy set *A* is equal to 1 for the entire real line, with the membership values counted from the membership function of *A*.

This axiom states that if $\mu_A(x)$ is the membership function of a fuzzy set *A*, then $(1 - \mu_A(x))$ would give us the membership values for any *x* of the complement. The membership function of the complement is a constant, equal to 1, but it has to be counted from $\mu_A(x)$, the membership function of *A*.

3. Discussions

The first axiom states that two probability measures can describe the membership function of a normal fuzzy number. This axiom gives us a way to construct normal fuzzy numbers. It does not actually defy the Dubois – Prade definition of a normal fuzzy number. The distribution function of a random variable is non-decreasing, and the complementary distribution function of a random variable is non-increasing.

The attempts to frame a law of probability in an interval on which a law of fuzziness was already defined, was defective. We need two laws of randomness in two intervals [*a*, *b*] and [*b*, *c*] to define a normal fuzzy number [*a*, *b*, *c*], and not one just law of *probability* in the entire interval [*a*, *c*]. The earlier attempts to link probability with fuzziness failed due to this reason.

Axiom – 1 explains why the triangular fuzzy number is the simplest among the fuzzy numbers. It is because the two random variables defining the left and the right Dubois-Prade functions are uniformly distributed. The uniform law being the simplest law of randomness, every triangular number is the simplest of all fuzzy numbers. The minimum temperature in any place is a probabilistic random variable, so is the maximum temperature. Therefore the daily temperature in a particular place is fuzzy, with the left reference function defined by the probability distribution function of one random variable and the right reference function defined by the complementary probability distribution function of another. When we assume that a fuzzy number is triangular, we actually assume that the two density functions concerned are constants; this means that the two laws of randomness are uniform and deterministic, there is no probability associated with them.

The second axiom straightaway leads to the facts that the union of a normal fuzzy set and its complement is indeed the universal set, and that the intersection of a normal fuzzy set and its complement is the null set. Accordingly, fuzzy sets can be seen to form a field. If $3/4^{th}$ of the height of a glass is filled with water, the empty portion, $1/4^{th}$ of the total height, is to be counted not from level zero, but from the level upto which there is water.

1/4th of the total height is the value of level of absence of water, and this is to be counted not from level zero, but from the level upto which water is there. According to the definition forwarded by Zadeh, a fuzzy set can actually include its complement. In fact, if for a fuzzy set *A*, the membership function is a constant equal to ¾, then by the Zadehian definition the complement A^C would have membership function equal to ¼, and thus in this case *A* includes A^C. Is that kind of a picture evident from our example of the glass of water mentioned above? Clearly, there was a fault in describing the membership function of the complement of a fuzzy set; what we have proposed as Axiom – 2 removes that.

Baruah [4] has shown that two probability measures are necessary and sufficient to define a law of normal fuzziness. That result has been stated as Axiom- 1 in the present article. The second axiom too was discussed therein. In the present article, we have just presented that fuzziness too, like probability, can be defined axiomatically; it need not be believed to be heuristic. We would like to repeat that Axiom – 1 does not defy the Dubois-Prade definition of a normal fuzzy set. Axiom – 2 follows naturally from the fact that not everything can be counted from the zero level; otherwise the six feet tall man of our example standing on a three feet high table could claim to be nine feet tall. If fuzzy sets are defined using these two axioms, it is obvious that the theory of fuzzy sets can be seen to be a natural extension of the classical theory of sets, which should naturally be the case.

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