Cardinality of Fuzzy Sets: An Overview

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Abstract

In this article, we would like to stress on the new definition of cardinality of fuzzy sets which can be contributed to the definition of complementation of fuzzy sets on the basis of reference function. As a consequence of which we would like to discard those results which are based on the existing definition of cardinality of fuzzy sets particularly when it involves complementation.

Keywords: *reference function, membership function, complementation of fuzzy sets*

1. Introduction

Since fuzzy set theory proposed by Zadeh, it has been developed in theory and applications in the past 45 years. In fuzzy set theory, we can see the use of the term cardinality which is most commonly used concept in many areas. Cardinality belongs to most important and elementary characteristics of a set. The cardinality of a crisp set is the number of elements in the set. Using fuzzy sets which are many-valued generalization of sets, one likes to have for them analogus characteristics and in the process different approaches can be found in the fuzzy literature. The main development of the approaches to the cardinality of fuzzy sets proceeded in the following two directions.

In fuzzy mathematics, the cardinality of fuzzy sets is a measure of the number of the elements belonging to the set. Analogously to the cardinal theory of sets, there are two approaches to the cardinal theory of fuzzy sets- one of which is the fuzzy cardinality of fuzzy sets and the other which uses ordinary cardinal, ordinal numbers or real numbers or some generalization of ordinal numbers or cardinal numbers. In the first approach, since the results based on comparision of two fuzzy sets are rather theoritical than practical, the main attention to the fuzzy cardinal theory was focused on the second approach.

In scalar approaches, the cardinalities of fuzzy sets were defined with the help of a mapping that to each (mainly finite) fuzzy set, assigns a single ordinary cardinal number or a non-negative real number. It is important to note here that a finite fuzzy set is understood as a fuzzy set having a finite support. The scalar cardinality was proposed by De Luca and Termini who named this as the power of a finite fuzzy set. The power of a finite fuzzy set A is given by sum of the membership degrees of the fuzzy set A. Accordingly, the scalar cardinality of fuzzy set

$$A: \Omega \rightarrow [0,1]$$

is defined as the sum of the membership degrees of finite fuzzy set A. That is symbolocally defined as

$$|A| = \sum_{x \epsilon \Omega} A(x)$$

Since an element can partially belong to a fuzzy set, a natural generalization of the classical notion of cardinality is to weigh each element by its membership degree, which resulted in the following formula for cardinality of a fuzzy set:

$$|A| = \sum \mu_A(x), \forall x \epsilon \Omega$$

this $|A|$ is called the sigma- count of A.

Other definitions of cardinalities as well as their properties can be found in [15 &16].

Zadeh introduced a relative measure of scalar cardinality of fuzzy sets by:

$$\sum count(\frac{A}{B})$$

That is to say that the cardinality of a fuzzy set A, the so called sigma- count, is expressed as the sum of the membership values of A. This approach to cardinality of fuzzy sets is convenient in applications and therefore favoured by many practioners. However there are many approaches to this evaluation. The problem of counting fuzzy sets has generated a lot of literature since Zadeh's initial conception. It is most widely used concept in fuzzy areas since it is useful in answering many questions. Therefore, it plays an important role in fuzzy databases and information systems. Other definition of scalar cardinalities and their properties can be found in literature. Moreover, an axiomatic approach to scalar cardinalities of finite fuzzy sets was studied by Wygralak in [17]. Some other relationship between fuzzy mappings and scalar cardinalities can be found in [18, 19, 20 and 21].

On the other hand, fuzzy cardinality of fuzzy sets is itself also a fuzzy set on the universe of natural numbers.The first definition of fuzzy cardinality of fuzzy sets by means of functions from N to [0,1] was done due to Zadeh and it is based on alpha cuts of fuzzy sets. Here in this article we would like to deal with the scalar cardinalities only.

We would like to refute those papers with the introduction of the new definition in which the traditional definition of cardinality was used especially with the complementation of fuzzy sets. It is important to note here that in case of usual fuzzy sets, the existing definition of cardinality can be used without having any serious impact on the results obtained but while dealing with the complementation care should be taken because otherwise we would have to be satisfied with an unreliable outcome.

There are many other areas of fuzzy sets in which we can find the influence of fuzzy cardinality. These are too numerous to mention in this article but in order to prevent any confusion a few results which involve cardinality to some extent are reviewed for illustration purposes in the next section.

2. Some Papers Related to Fuzzy Cardinality

Janssen, Baets and Meyer [22], proposed a family of fuzzification scemes for transforming cardinality-based similarity and inclusion measures for ordinary sets into similarity and inclusion measures for fuzzy sets in finite universe. The family they found was based on the rule for fuzzy sets cardinality and for standard operation on fuzzy sets. These are written in the following manner:

Inclusion measures:

$$I(A,B) = \frac{x(b-u) + \acute{x}(a-u) + yu + x(n-a-b+u)}{\acute{x}(a+b-2u) + \ yu + z(n-a-b+u)}$$

the parameters x, y, z and $\acute{x}$ are non negative and reals.

Similarity measures:

$$S(A,B) = \frac{x(a+b-2u) + yu + x(n-a-b+u)}{\acute{x}(a+b-2u) + \ yu + z(n-a-b+u)}$$

where

$$a = card(A) = \sum a_i, \text{ i=1, 2, 3, n}$$

$$b = card(B) = \sum b_i \text{ and } c = card(C) = \sum c_i$$

$$u = \sum T(\ \ a_i, b_i) \text{ and } T(\ \ a_i, b_i) = A \cap B(x_i)$$

Bason, Neagu and Ridley [21] introduced an equation to measure similarity between rwo fuzzy sets based on fuzzy set theoritical concepts by using cadinality of fuzzy sets and their operations. It was written in the following form:

$$S(A,B) = \frac{\alpha|A \cap B|}{\alpha|A \cap B| + \beta|A - B| + \gamma|B - A|}$$

for $\alpha > 0$ and $\beta, \gamma \geq 0$

where$|A| = card(A) = \sum \mu_A\,(x_i)$, $\forall x \epsilon \Omega$

It was further mentioned that the operation between fuzzy sets $A \cap B$ and $A \cup B$ were performed in terms of membership functions using Zadeh's definition of union and intersection.The same was followed in case of complementation. It is important to mention here that according to Zadehian definition,

$$A - B = A \cap B^c$$

and hence

$$|A - B| = |A \cap B^c|$$

indicating that $|A-B|$ involves complementation of fuzzy sets and same is for $|B-A|$. But since there are some shortcomings in the existing definition of complementation of fuzzy sets which are discussed in details in our previous works and hence we would not like to continue with the defintion of cardinality in case of complementation of fuzzy sets. This reason has led us to define it in accordance with the definition of complementation as defined by Baruah in his works which can be found in [6]. This finding was successfully used in various works of Dhar [9, 10, 11, 12, 13&14]. But before proceeding further we would like to mention in brief about the new definition of complementation of fuzzy sets defined by Baruah [6].

3. New Definition of Complementation of Fuzzy Sets

Baruah [5, 6, 7, 8] has defined a fuzzy number N with the help of two functions: a fuzzy membership function $\mu_2(x)$ and a reference function $\mu_1(x)$ such that $0 \leq \mu_1(x) \leq \mu_2(x) \leq 1$ Then for a fuzzy number denoted by {x, $\mu_1(x), \mu_2(x), x\epsilon\Omega$} we would call $\{\mu_2(x)-\mu_1(x)\}$ as the fuzzy membership value, which is different from fuzzy membership function. This definition of complementation plays a vital role in introducing the following definition of cardinality of finite fuzzy sets.

4. New Definition of Cardinality of Fuzzy Sets

It is important to mention here that since we would like to define fuzzy sets with the help of two functions such as fuzzy membership function and fuzzy membership value. In parallel with what had been done for cardinalities of fuzzy sets, we shall define the cardinality of a fuzzy set A as:

$$|A| = card(A) = \sum\{\mu_2(x)-\mu_1(x)\}$$

It is important to mention here that for fuzzy sets expressed in the form

$$A = \{x, \mu_A(x), 0, x\epsilon\Omega\}$$

The cardinality would be the following

$$\sum(\mu_A(x)-0)$$

Similarly, the cardinality of the complement of the fuzzy set A expressed in the form

$$A^c = \{x, 1, \mu_A(x), x\epsilon\Omega\}$$

Would be

$$\sum(1-\mu_A(x))$$

Thus the main contribution of this article is to put forward a new definition of cardinality of fuzzy sets on the basis of reference function. That is to say that the new definition of cardinality is the result of the new definition of complementation of fuzzy sets.

5. Numerical Examples

Let us consider a fuzzy set in the usual case as

A= {(1, .2), (2, .4), (3, .6), (4, .8), (5, 1)}

and the complement of the set A, according to the existing definition is

$$A^c = \{(1,.8),(2,.6),(3,.4),(4,.2),(5,0)\}$$

Now the cardinality of the fuzzy set A^c, in accordance with the existing definition would be

$$|A^c| = .8+.6+.4+.2+0 = 2$$

Then this set would take the following form if the new definition of fuzzy sets based on reference function is taken into consideration

A= {(1, .2, 0), (2, .4, 0), (3, .6, 0), (4, .8, 0), (5, 1, 0)}

Now the complement of this set would take the following form

A^c = {(1, 1, .2), (2, 1, .4), (3, 1, .6), (4, 1, .8), (5,1, 1)}

According to the proposed definition of cardinality, the cardinality of the set A^c will be calculated as

$$|A^c| = (1-.2)+(1-.4)+(1-.6)+(1-.8)+(1-1)$$

= .8+.6+.4+.2+0

= 2

The result obtained will coincide with the cardinality of A^c if it were calculated by using existing definition. But it is important to mention here that the result may vary in some cases which especially involve union and intersection of a fuzzy set and its complement set. Let us see how it looks from the following examples:

Let consider the following fuzzy sets in accordance with reference function

A= {(1, .2, 0), (2, .5, 0), (3, .8, 0), (4, 1, 0), (5, .7, 0), (6, .3, 0)}

and the complement A^c of A in the following manner

$$A^c = \{(1,1,.2),(2,1,.5),(3,1,.8),$$

$$(4,1,1),(5,1,.7),(6,1,.3)\}$$

Then we get

$$A \cup A^c = \{(1,1,0),(2,1,0),(3,1,0),(4,1,0),(5,1,0),(6,1,0)\}$$

which is the universal set Ω.

Thus the cardinality of the fuzzy set $A \cup A^c$ would be equal to the cardinality of the universal set Ω which is equal to the number of elements of the set.

Again we have

$$A \cap A^c = \{(1, .2, .2), (2, .5, .5), 3, .8, .8), (4,1,1), (5, .7, .7)$$
$$(6, .3, .3)\}$$

which is the null set $\emptyset$.

Hence the cardinality of $A \cap A^c$ will be equal to zero according to our new proposal.

But the existing definition would produce a completely different result. Thus here we can observe the difference that exists between the current definition of cardinality of fuzzy sets and the newly introduced definition of fuzzy sets on the basis of reference function.

Again with the introduction of the new definition of cardinality of fuzzy sets, we would like to mention that this definition would also satisfy the propositions set by Zadeh and Dubios and Prade. The propositions are as follows:

Proposition 1: Zadeh [3]

Let A and b be two fuzzy sets on U, then

$$\max(|A|, |B|) \leq |A \cup B| \leq |A| + |B|$$
$$\max(0, |A| + |B|) \leq |A \cap B| \leq \min(|A|, |B|)$$

Proposition 2: Dubois and Prade [18]

Let A and B be two fuzzy sets on universe U, then

(i) If$A \subseteq B$, then |A|$\leq |B|$, where $A \subseteq B$ is defined as $\forall x \epsilon U$, A(x) $\leq$B(x).
(ii) $|A^c| = |U|$ - $|A|$ (when U is finite)
(iii) $|A \cup B|$ + $|A \cap B|$=$|A| + |B|$

6. Conclusions

In this article, we intended to revisit the existing definition of cardinality of fuzzy sets and in the process it is found that the cardinality of a fuzzy set especially when dealing with complementation is not defined logically. The reason behind such a claim is contributed to the fact that the existing definition of complementation is not logically defined. We have explained the meaning as well as some motivations. Hence it is obvious that any result which is obtained with the help of something which itself is controversial cannot yield a suitable result. It is observed that the complementation defined with the help of reference function seems more logical than the existing one. If this be the case, then there would be problem in finding the cardinality of such a set. It is due to this reason; we would like to propose a new definition of cardinality of fuzzy sets on the basis of membership value. Further, it can be seen that the propositions which were established by previous researchers are also satisfied by the new formulation.

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