A Note on Subsethood Measure of Fuzzy Sets

Mamoni Dhar

Assistant Professor, Department Of Mathematics
Science College, Kokrajhar-783370, Assam, India
mamonidhar@rediffmail.com, mamonidhar@gmail.com

Abstract

The main aim of this article is to revisit the definitions of subsethood measure which are predomonant in the literature of fuzzy set theory. It is also intended to revisit the relationship between fuzzy entropy and conditioning as proposed by some authors. We wish to analyze these with the extended definition of complementation of fuzzy sets on the basis of reference function. The existing definition of cardinality of fuzzy sets should have to be changed accordingly and a new definition is proposed hereby. Here efforts have been made to show that those proposed definitions of subsethood measure are not acceptable. Again due to the same reason we would like to mention here that the entropy-subsethood relationship which had been introduced by some authors is also not logical.

Keywords: *fuzzy membership value, fuzzy membership function, superimposition of sets, Shannon's indices*

1. Introduction

Subsethood measure is an important concept of fuzzy set theory. In the literature, there are several well known measures of subsethood. It stands for the degree that a given fuzzy set B is a subset of another set A. Many contributions on the measure of the degree of subsethood between two fuzzy sets have already been made. But most commonly used measure of subsethood is the one proposed by Kosko [2]. The fuzzy subsethood theorem was derived both algebraically and geometrically by Kosko who designated it the fundamental unifying structure in fuzzy set theory. Given two fuzzy sets A and B, it defines the degree to which one set belongs to another and it is expressed by the formula

$$S(A,B) = \frac{M(A \cap B)}{M(A)}$$

where M(A) represents the fuzzy cardinality of fuzzy set A.

It can be defined in another form which is as follows:

For any pair of fuzzy subsets defined on a finite universal set Ω, the degree of subsethood S (A, B) of A in B is defined by the formula:

$$S(A,B) = \frac{1}{|A|}\{|A| - \sum_{x \epsilon X} max\,\{0, (A(x) - B(x)\}\}$$

$$\text{where } |A| = \textstyle\sum_{x\epsilon X} A(x), \qquad x\epsilon\Omega$$

A measure of fuzziness which is often used in the literature of fuzzy information is entropy, first mentioned by Zadeh [18]. Several techniques have been in use to measure the entropy.

Kosko [1], proposed a non-probabilistic entropy measure in the context of fuzzy sets and found the fuzzy entropy measure of a fuzzy set is the degree to which $A \cup A^c$is a subset of $A \cap A^c$. In other words, to relate subsethood measure with fuzzy entropy the following expression was proposed.

$$R_1(P) = S(A \cup A^c, A \cap A^c)$$

where $R_1(P) = \frac{\sum count(A \cap A^c)}{\sum count(A \cup A^c)}$ and

$S(A \cup A^c, A \cap A^c)$ measures the degree of subsethood of $A \cup A^c$ in $A \cap A^c$ and A^c is the complement of the fuzzy set A ,the membership function of which is considered as one minus the membership function of the fuzy set A.

In other words, it can be interpreted that $R_1(P)$ represents the degree to which the superset $A \cup A^c$ is a subset of its own subset $A \cap A^c$. That is to say, by using the aforesaid formula of subsethood we can have to a certain degree, the universe of discourse is also contained in any of its subsets. So here is confusion how a superset can be a subset of its own subset. This is not desirable. But according to fuzzy set theory this is not surprising since fuzzy sets violates by definition, the two basic properties of complement of crisp sets, the law of contradiction and the law of excluded middle.

Accordingly, entropy of fuzzy sets was defined. Then it was assumed that there exist a close connection between entropy and conditioning which has nothing to do with probability theory. On this basis, the theory of subsethood was then used to solve one of the major problems with Bayes theorem learning and its variants-the problem requiring that the space of alternatives be portioned into disjoint exhaustive hypothesis.

On the basis of Kosko's subsethood measure, fuzzy entropy and Willmot's work (see for example [17]), Young [16] defined subsethood measure and weak subsethood measures which are as follows:

A real function

$$c: F(\Omega) \times F(\Omega) :\rightarrow [0, 1]$$

is called a subsethood measure if c has the following properties:

i. C(A,B)=1 if and only if $A \subset B$, i.e. ,μ_A(x)≤ μ_B(x), $\forall x \epsilon \Omega$

ii. If $[\frac{1}{2}] \subset A$, then $c(A, A^c) = 0$ if and only if A=X.

iii. If $A \subset B \subset C$, then c(C,A) ≤c(B,A) and if $A \subset B$, then c(C,A) ≤c(C,B)

Young had pointed out that there exist some formulas which donot satisfy the first condition but satisfy ii & iii. In such cases c is called a weak subsethood measure. It has to be pointed out that the third condition stemmed from Kosko's subsethood measure.

But Fan, Xie, Pie [15] were not satisfied with the Young's definition of subsethood since they were of the opinion that axiom definition should be abstract and simple.They tried to modify this and in the process, they found some new definitions of subsethood measures as "* subsethood measures", "strong subsethood measures" and "weak subsethood measures".Let us have a lo0k at these definitions.

1.1. Strong Subsethood Measure:

A real function $c: F(\Omega) \times F(\Omega): \rightarrow [0, 1]$

Is called a strong subsethood measure if c has the following properties:

i. if $A \subset B$, then C(A,B)=1

ii. if $\mathrm{A} \neq 0$ and $\mathrm{A} \cap \mathrm{B} = 0$, then $\mathrm{c(A, B)} = 0$

iii. If $A \subset B \subset C$, then c(C,A) ≤c(B,A) and c(C,A) ≤c(C,B)

1.2. Subsethood Measures:

A real function $c: F(\Omega) \times F(\Omega): \rightarrow [0, 1]$

is called a subsethood measure if c has the following properties:

i. if $A \subset B$, then C(A,B)=1

ii. $\mathrm{c(X, 0)} = 0$

iii. If $A \subset B \subset C$, then c(C,A) ≤c(B,A) and c(C,A) ≤c(C,B)

1.3. Weak Subsethood Measure:

A real function $c: F(\Omega) \times F(\Omega): \rightarrow [0, 1]$

is called a weak subsethood measure if c has the following properties:

i. if C(0,0)=1, c(0, Ω)=1 and c (Ω, Ω)=1;

ii. $\mathrm{c}(\Omega, 0) = 0$;

iii. If $A \subset B \subset C$, then c(C,A) ≤c(B,A) and c(C,A) ≤c(C,B);

Accordingly, they defined functions for finite set X as

$$C_j(A,B) = \begin{cases} 1, \quad A = 0, or, B = \Omega \\ \dfrac{M(A \cap A^c \cap B \cap B^c)}{M(A \cap B^c)}, \qquad otherwise \end{cases}$$

and

$$C_G(A,B) = \begin{cases} 1, \quad A = 0, or, B = \Omega \\ \dfrac{M(A \cap A^c \cap B \cap B^c)}{min(M(A), M(B^c))}, \qquad otherwise \end{cases}$$

And found C_j as the * subsethood measures and C_G as a weak subset hood measures. So here, we can see the use of cardinalities with the complement of fuzzy sets to a great extent in defining subsethood measures.

It is to be mentioned here that these results were obtained on the basis of defining complement of a set as one minus the membership function of the given set. In this article, our main aim is to show that this proposed definition of entropy is unable to give us the desired result for which it was introduced and it is not reliable and with this definition it is not possible to find a link of it with subsethood in the said manner with the help of definition of complementation of fuzzy sets based on reference function as proposed by Baruah [3, 6]. Furthermore, we shall show that there exist a link between possibility and probability and hence the other assumption that entropy and conditioning has nothing to do with probability can also be nullified. Moreover, we would like to say that those definitions of subsethood measures, strong subsethood measures and weak subsethood measures have no meaning from our standpoints and these can be seen from the examples mentioned in their articles. It is to be noted here that there are many authors who have some doubts about Zadeh's definition of fuzzy set theory and they tried to define it according to their view point but a few of them were interested in defining complement of a fuzzy set differently.

In this article, we are interested in dealing with the way of defining complementation which is different from that of Zadeh and finally we would like to say that unlike classical set, fuzzy sets also satisfy excluded middle laws and hence the aforesaid definitions would not give us logical results.We would like to mention one such paper, the author of which unlike us was not satified with Zadehian definition of complementation.

2. Some Papers Dealing with Complementation

Gao, Gao and Hu [14] found that there are some mistakes Zadeh's fuzzy sets and found that it is incorrect to define the set complement as $\mu_{A^c}(\mathbf{x}) = 1 - \mu_A(\mathbf{x})$, because it can be proved that set complement may not exist in Zadeh's fuzzy set theory. According to them it leads to logical confusion, and is seriously mistaken to believe that logics of fuzzy sets necessarily go against classical and normal thinking, logic, and concepts.Since they found some shortcomings in the Zadeh's fuzzy set theory ,they wanted to move away from it and worked towards removing the shortcomings which according to them debarred fuzzy sets to satisfy all the properties of classical sets.They introduced a new fuzzy set theory,called *C*-fuzzy set theory which satisfies all the formulas of the classical set theory. The *C*-fuzzy set theory proposed by them was shown to overcome all of the errors and shortcomings, and more reasonably reflects fuzzy phenomenon in the natural world. It satisfies all relations, formulas, and operations of the classical set theory. Here we can find one such paper in which the authors are not satisfied with the way in which complementation is defined. We do agree with the authors of this paper. But we would like to proceed differently. In order to establish our claim, we would like to depend on the following definition only.

3. Baruah's Definition of Complement of a Fuzzy Set

Baruah [3 & 6] has defined a fuzzy number N with the help of two functions : a fuzzy membership function $\mu_2(x)$ and a reference function $\mu_1(x)$ such that $0 \leq \mu_1(x) \leq \mu_2(x) \leq 1$ Then for a fuzzy number denoted by $\{x, \mu_1(x), \mu_2(x)\}$ we would call $\{\mu_2(x) - \mu_1(x)\}$ as the fuzzy membership value ,which is different from fuzzy membership function .It is to be noted here that in the definition of complement of a fuzzy set, fuzzy membership value and the fuzzy membership function have to be different, in the sense that for a usual fuzzy set the membership value and membership function are of course equivalent.

In accordance with the process discussed above, a fuzzy set defined by

$$A = \{x, \mu(x), x\epsilon\Omega\}$$

would be defined in this way as

$$A = \{x, \mu(x), 0, x\epsilon\Omega\}$$

so that the complement would become

$$A^c = \{x, 1, \mu(x), x\epsilon\Omega\}$$

The extended definition using a reference function leads to the assertion that for any fuzzy set A, we must have

$A \cap A^c =$ the null set φ and

$A \cup A^c =$ the universal set Ω

In other words the two laws which were assumed to be true only for classical sets hold for fuzzy sets also. Consequently the definition of cardinality of a fuzzy set which involves complementation should have to be changed to keep space with the new definition. The following is an effort to define it accordingly.

4. Cardinality of a fuzzy Set

The cardinality of fuzzy sets which was previously defined as the sigma count of the membership function of the set would have to be defined by sigma count of the membership values according to the new definition of complementation of fuzzy sets. It is important to mention here that in case of usual fuzzy sets, the existing definition would work but for the complement of a fuzzy set, the proposed definition should have to be considered to get a suitable result.

Hence, it is important to mention here the fact that the definition of complementation of fuzzy sets and that of cardinality sholud should be taken care of in order to get a suitable result. If these definitions are considered then the fuzzy entropy measure $R_1(P)$ defined above will always yield the result zero, Dhar.et.al [9]. Now if $R_1(P) = 0$ then $S(A \cup A^c, A \cap A^c) = 0$ and hence we shall never get the degree of subsethood of $A \cup A^c$ in $A \cap A^c$. Again from the above results, we can see that the other examples like $C_j(A, B)$ and $C_G(A, B)$ which were proved to be * subsethood measure and weak subsethood measure respectively with the axiom definitions of subsethood would not give any subsethood measure at all.

Hence it can be said that the definition of entropy introduced in that manner fails to give us desired result. Another thing to be noticed here that since the definition of entropy is defective and hence it cannot be associated with conditioning. Further it is to be mentioned that since the proposed relationship is not a reliable one so it cannot be used to solve problems in any other fields and hence it has nothing to do in solving any problem of Bayes theorem.

In the same way, the entropy subsethood relationship formulated in that manner seems to be unworkable and so unacceptable. Another thing to be worth mentioning here that there exists a close link between possibility and probability and hence the link between probability and fuzzy entropy cannot be overlooked, which will be clear with the following few lines:

With the help of operation of superimposition of sets, Baruah [4, 5& 6], found that a normal fuzzy number N= $[\alpha, \beta, \gamma]$ defined with a membership function $\mu_N(x)$, where

$$\mu_N(x) = \psi_1(x), \text{ if } \alpha \le x \le \beta$$
$$= \psi_2(x) \text{ , if } \beta \le x \le \gamma$$
$$= 0, \text{ otherwise.}$$

with $\psi_1(\alpha) = \psi_2(\gamma) = 0$,$\psi_1(\beta) = \psi_2(\beta) = 1$, the partial presence of a value x of a variable X in the interval $[\alpha, \gamma]$ is expressible as:

$$\mu_N(x) = \theta \text{ Prob } [\alpha \le X \le x] + (1-\theta) \{1\text{-Prob}[\beta \le X \le x]\}$$

where if $\theta = 1, \alpha \le x \le \beta,$ if $\theta = 0,\ \beta \le x \le \gamma.$

So, $\mu_N(x)$ for $\alpha \le x \le \gamma$ is nothing but a probability density only, which can be expressed either as $Prob[\alpha \le X \le x]$or as $\{1 - Prob[\beta \le X \le x]\}$ whichever is the case.

This definition takes into account the fact that the membership function explaining a fuzzy variable taking a particular value is either the distribution function of a random event or the complementary distribution function of another random event. It was thus established that two laws of randomness are needed to define one possibility law. In other words, not one but two distributions with reference to two laws of randomness defined on two disjoint spaces can construct a fuzzy membership function. Here we can see that possibility is related to probability. Using Shannon's Indices on the suggested link between probability and possibility, we can get a pair of entropies for a normal fuzzy number, Dhar .et.al [9 & 11] Thus the statement that entropy has nothing to do with probability is discarded.

5. Conclusions

In this article, different measures of subsethood as can be found in the literature of fuzzy set theory are revisited. Further the relationship between entropy and subsethood has been discussed and commented on. All these are analyzed with the help of the definition of complementation of fuzzy sets on the basis of reference function. Since fuzzy entropy always becomes zero when the suggested definition of complementation is used, it says that this type of formulation is not workable. It is for this reason, the link between entropy and subsethood theory cannot be formulated in the manner proposed by some authors. Moreover, the other axiom definitions which are used to test the subsethood measures in different categories are also not dependable due to the same reason. Furthermore, it is observed that the existing definition of cardinality is not sufficient to find the cardinality of complementation of fuzzy sets, if it is defined on the basis of reference function and so the need for new definition of the cardinality of a fuzzy set in case of complementation is highlighted in this article. So some further researches are required in finding subsethood measure of fuzzy sets. Moreover, the link between probability and possibility has been discussed as an indication of the fact that there is a link between probability and posibility, which in turn helps in finding entropy of fuzzy sets.

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