A New Approach to Visualization of Fuzzy Set Operations

Samarjit Das

Department of Computer Science & IT, Cotton College, Guwahati-01, Assam, India
ssaimm@rediffmail.com

Abstract

The elements in fuzzy set and standard fuzzy set operations like union, intersection and complement of a fuzzy set have been tried to be visualized with diagram, known as disk diagram. The existing design of the disk diagram is based on the traditional Zadehian theory of fuzzy sets where it is believed that there is no difference between fuzzy membership function and fuzzy membership value for the complement of a fuzzy set which is already proved to be wrong and as a consequence of which it could not visualize properly the complement of a fuzzy set. It also failed to visualize the operations like union, intersection of a fuzzy set with its complement. We design a system to visualize the elements in fuzzy set and standard fuzzy set operations like union and intersection. Our design emphasizes in visualizing the complement of a fuzzy set as there is ambiguity in the existing design while visualizing the complement of a fuzzy set. Our present design is based on the fact that fuzzy membership function and fuzzy membership value for the complement of a fuzzy set are two different things. With this new approach it has been tried to visualize the fuzzy set operations like union, intersection, specially the complement of a fuzzy set in a proper way.

Keywords: *Disk diagram, membership value, membership function, complement of a fuzzy set*

1. Introduction

With the help of disk diagram [1] it has been tried to visualize [2] the elements of a fuzzy set and the standard fuzzy set operations like union, intersection and complement of a fuzzy set. The entire design of the existing disk diagram [2] is based on the traditional Zadehian theory of fuzzy sets where it is believed that there is no difference between fuzzy membership function and fuzzy membership value for the complement of a fuzzy set which is already proved to be wrong [3] and as a consequence of which it could not visualize properly the complement of a fuzzy set. As there is ambiguity in the concept of the complement of a fuzzy set in Zadehian theory of fuzzy sets, so in the belief that if A is a fuzzy set then neither $A \cap A^c$ is null set nor $A \cup A^c$ is the universal set [3, 5] and therefore the existing design [2] failed to visualize $A \cap A^c$ as a null set and $A \cup A^c$ as a universal set. Here we put forward a new design which is based on the fact that the fuzzy membership function and fuzzy membership value are two different things for the complement of a fuzzy set [3]. Each element in a fuzzy set is associated with a fuzzy membership value which can further be expressed as a difference of a fuzzy membership function and a function of reference [3, 4] i.e. if x is a fuzzy number then its membership value $\mu(x)$ is expressed as

$\mu(x) = \mu_1(x) - \mu_2(x) \rightarrow (1)$, where $\mu_1(x)$ and $\mu_2(x)$ are membership function and function of reference respectively.

Now, if we consider the function of reference, $\mu_2(x) = 0$ in (1), then we get $\mu(x) = \mu_1(x)$ i.e. there is no difference between the membership value and the membership function of a fuzzy number x. But the fact is that we can't always consider $\mu_2(x) = 0$, particularly in the case of the complement of a fuzzy set. For A^c the membership function is 1 everywhere with the reference function being $\mu(x)$, while for A the membership function is $\mu(x)$ with reference function being 0 everywhere [3]. In the existing approach [2] the membership function for the complement of fuzzy set is defined as

$\mu_2(x) = 1 - \mu_1(x) \rightarrow (2)$, where $\mu_1(x)$ and $\mu_2(x)$ are membership functions of a fuzzy set A (say) and its complement A^c respectively.

Now, if we consider a fuzzy element x with membership function $\mu_1(x)$, such that $\mu_1(x) = 0.5$ then from equation (2) we get $\mu_1(x) = \mu_2(x)$ i.e. there is no difference between the membership function of A and A^c and which has been already proved to be wrong [3]. Therefore the existing design failed to visualize properly the complement of a fuzzy set A(say) and as a consequent of which it could not visualize $A \cap A^c$ as the null set and $A \cup A^c$ as the universal set. In this paper through our design we will try to visualize elements of a fuzzy set, standard fuzzy set operations like union, intersection of two different fuzzy sets. With this new approach it has been tried to highlight the limitations of the existing design [2], specially in the case of a complement of a fuzzy set, and hence to minimize these limitations.

2. The Design

Different fuzzy data factors like data as elements of a set, individual membership function and function of reference of single data as element, membership value, frequency of data on the same membership value, a set itself along with their corresponding visual features of our disk diagram are as listed in Table 1.

Table 1. Fuzzy Data Factors and Corresponding Visual Features in Disk Diagram

Fuzzy data factors	Corresponding visual features in a disk
Data	Dot
Membership function	Ring with dot/s
Reference function	Ring without a dot
Membership value	Horizontal line
Frequency of data	Bar
Set	Disk

A sample data set is listed in Table 2 to explain the visual feature of our design. In Table 2 there are two fuzzy sets A and B and each data in a set is associated with its corresponding membership function between 0 (0%) and 1(100%). In our design we define a fuzzy set A (say) as

$A = \{x, \mu_1(x), \mu_2(x); x \in \Omega\}$, where x is an element in set A , $\mu_1(x)$ and $\mu_2(x)$ are the membership function and function of reference for the data x respectively and Ω is the universal set [3].

Now, to visualize a fuzzy set A (say) we draw a disc with radius equal to 1 unit centred at 0. As with each data a membership function, $\mu_1(x)$ and a function of reference, $\mu_2(x)$ are associated, therefore to visualize a data x in set A we draw two concentric rings – one with radius equal to $\mu_1(x)$ centred at 0 and the other with radius equal to $\mu_2(x)$ centred at 0 and put a dot anywhere on the ring whose radius is equal to $\mu_1(x)$. The shortest distance between the circumferences of the two concentric rings gives the corresponding membership value of x and is represented by a horizontal line. The number of dots on the same ring will give us the frequency of the corresponding data and is represented by a bar. So, for each data the system will produce two rings – one with dot/s for the membership function and the other without a dot for the function of reference. (Here for simplicity, except for the complement of a fuzzy set, we consider the reference function i.e. $\mu_2(x) = 0$ for each data in Table 2). In this case where $\mu_2(x) = 0$, the system will produce only one ring representing the membership function and as the radius of the reference function is zero, it will not be visible. As for example data1 in set A of Table 2 is visualized with a dot on the ring with centre at 0 and with a radius equal to .15. Similarly data3 in set A of Table 2 is visualized with a dot on the ring with centre at 0 and with radius equal to .76. Here the reference function, $\mu_2(x) = 0$ is not visible as its radius is equal to zero. The ring with radius equal to $\mu_1(x)$ centred at 0, represents the membership function and the radius of the ring (which represents the membership function) represents the membership value of the fuzzy element x.

Table 2. Data as Fuzzy Elements with Membership Function in set A and B

Data	Set A	Set B
Data1	15%	58%
Data2	15%	0%
Data3	76%	65%
Data4	23%	47%
Data5	36%	47%
Data6	43%	67%
Data7	76%	19%
Data8	15%	43%

As the maximum membership value is 1 for a data in a fuzzy set, therefore the radius of any ring can't be more than 1 with centre at 0. Therefore the whole fuzzy set (say A) is represented by a disk with centre at 0 and with a radius equal to 1. With our design data in fuzzy sets A and B of Table 2 have been visualized in Figure 1 and Figure 2 respectively.

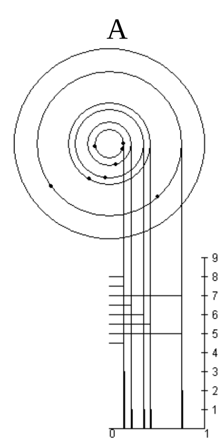

Figure 1. Visualization of Data in Fuzzy Set A

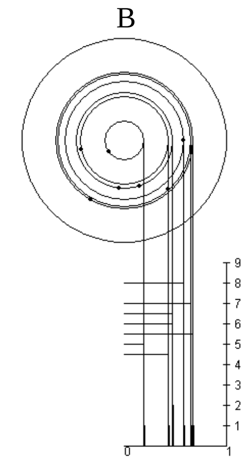

Figure 2. Visualization of Data in Fuzzy Set B

2.1. Set Operations

In the following sections the standard fuzzy set operations like union, intersection of two different fuzzy sets have been visualized with the help of our design.

2.1.1. Union of Two Different Fuzzy Sets: We define the fuzzy sets A and B as

A = {x, $\mu_A(x)$, 0; x∈Ω} and
B = {x, $\mu_B(x)$, 0; x∈Ω} , where $\mu_A(x)$, 0 are the membership function and function of reference of x in fuzzy set A, $\mu_B(x)$, 0 are the membership function and function of reference of x in fuzzy set B and Ω is the universal set. [3]

Now we define the union of A and B as

$$A \cup B = \{x, \max\{\mu_A(x), \mu_B(x)\}, \min\{0, 0\}; x \in \Omega\}$$
$$= \{x, \max\{\mu_A(x),\mu_B(x)\}, 0; x \in \Omega\}[3]$$

To visualize the elements in A∪B our system follows the following steps.

Step 1: draws disc A and B with radius 1 in each (as shown in Figure 3)

Step 2: finds out the max {$\mu_A(x)$,$\mu_B(x)$}

Step 3: if max {$\mu_A(x)$,$\mu_B(x)$}= $\mu_A(x)$

draws a ring with radius $\mu_A(x)$ concentric in disc A

else

draws a ring with radius $\mu_B(x)$ concentric in disc B

Step 4: puts a dot anywhere on the ring drawn in step3

Step 5: draws a tangent to the ring drawn in step3 to meet perpendicularly

the horizontal line segment below the corresponding disc (as shown in Figure 3.)

Step 6: draws a horizontal line segment equal to radius of the ring drawn in step3 anywhere on the tangent drawn in step5 (as shown in Figure 3)

Step 7: repeats step2 through step6 until all the elements are visualized

Step 8: counts the number of dots on a ring

Step 9: draws a bar leveling the value obtained in step8 on the corresponding tangent drawn in step5 to represent the frequency of data on the same ring (as shown in Figure 3).

The visual representation of union of fuzzy Sets A and B of Table 2 has been shown in Figure 3.

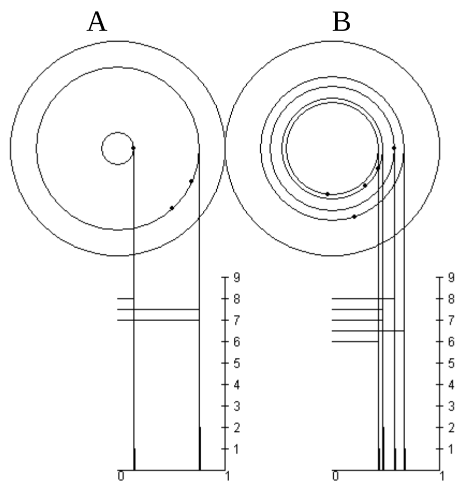

Figure 3. Visualization of Union of Fuzzy Sets A and B

2.1.2. Intersection of Two Different Fuzzy Sets: The standard fuzzy intersection of set A and set B (as defined in sec 2.1.1) is defined as

$A \cap B = \{x, \min\{\mu_A(x), \mu_B(x)\}, \max\{0, 0\}; x \in \Omega\}$
$= \{x, \min\{\mu_A(x), \mu_B(x)\}, 0; x \in \Omega\}$[3]

Like union, here also to visualize the elements in $A \cap B$ our system follows the following steps.

Step 1: draws disc A and B with radius 1 in each (as shown in Figure 4)

Step 2: finds out the $\min\{\mu_A(x), \mu_B(x)\}$

Step 3: if $\min\{\mu_A(x), \mu_B(x)\} = \mu_A(x)$

draws a ring with radius $\mu_A(x)$ concentric in disc A

else

draws a ring with radius $\mu_B(x)$ concentric in disc B

Step 4: puts a dot anywhere on the ring drawn in step3

Step 5: draws a tangent to the ring drawn in step3 to meet perpendicularly the horizontal line segment below the corresponding disc (as shown in Figure 4)

Step 6: draws a horizontal line segment equal to radius of the ring drawn in step3 anywhere on the tangent drawn in step5 (as shown in Figure 4)

Step 7: repeats step2 through step6 until all the elements are visualized

Step 8: counts the number of dots on a ring

Step 9: draws a bar leveling the value obtained in step8 on the tangent drawn in step5 to represent the frequency of data on the same ring. (as shown in Figure 4)

The visual representation of intersection of fuzzy Sets A and B of Table 2 has been shown in Figure 4.

3. Comparison of our Design with the Existing One

In this section we will highlight the limitations of the existing design [2] in visualizing the complement of a fuzzy set, the union of a fuzzy with its complement, the intersection of a fuzzy set with its complement etc. In the following sections we also discuss how our design is able to overcome the above limitations and succeeds in visualizing specially the complement of a fuzzy set, the union of a fuzzy with its complement, the intersection of a fuzzy set with its complement properly.

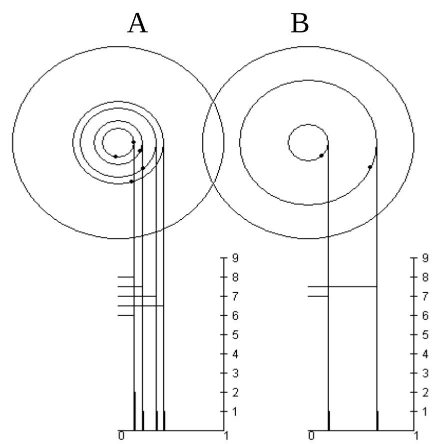

Figure 4. Visualization of Intersection of Fuzzy Sets A and B

3.1. Complement of a Fuzzy Set

The existing design [2] is based on Zadehian theory of fuzzy sets where it is believed that there is no difference between fuzzy membership function and fuzzy membership value for the complement of a fuzzy set. In Zadehian theory of fuzzy sets the membership function of complement A^c of a fuzzy set A is defined as 1- $\mu(x)$, where $\mu(x)$ is the membership function of A. This definition of membership function for the complement of a fuzzy set has already been proved to be wrong [3]. According to this new definition for A^c the membership function is 1 everywhere with the reference function being $\mu(x)$, while for A the membership function is $\mu(x)$ with reference function being 0 everywhere [3]. As the concept of reference function is absent in Zadehian theory of fuzzy sets, therefore it failed to find the difference between the membership function and membership value for the complement of a fuzzy set. As a consequence of which the existing design [2] could not visualize the complement of a fuzzy set properly. In Figure 5 visualization of fuzzy set A and its complement A^c, according to the existing design which is based on Zadehian theory of fuzzy sets has been shown. The modified form of visualization, put forward by our design, has been shown in Figure 6. In both the design the same set of data has been used from Table 2.

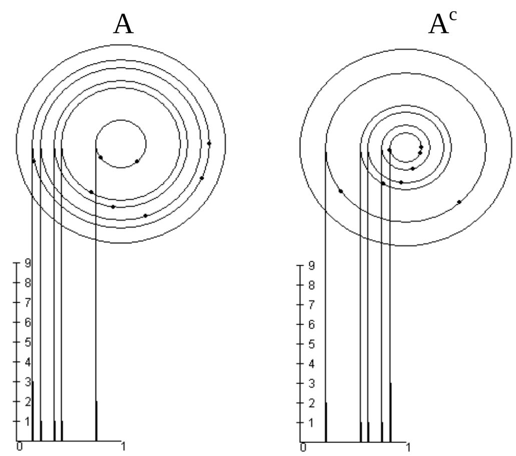

Figure 5. Visualization of Fuzzy Set A and Its Complement Ac According to the Existing Design

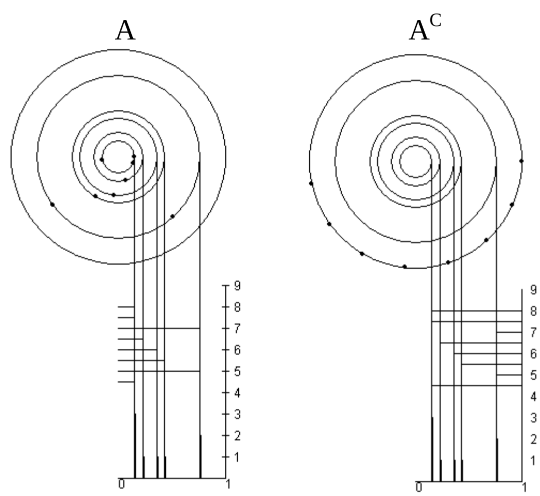

Figure 6. Visualization Fuzzy Set A and Its Complement A^c According to our Design

3.1.1. A Special Case: In this section the difference in visual representation produced by the existing design [2] and in that produced by our design for the complement of a fuzzy set has been shown by considering a special case where the membership function $\mu(x) = .5$ for a fuzzy element x. According to the existing design where the membership function for the complement of a fuzzy set is $1-\mu(x)$, there will not be any difference in the membership function of a fuzzy set and that of its complement set if $\mu(x)=.5$. Therefore the existing design produces identical visual representation of a fuzzy set and its complement set when the membership function $\mu(x) = .5$. Figure 7 shows the visualization, according to the existing design, of a fuzzy set and its complement set when the membership function $\mu(x) = .5$.

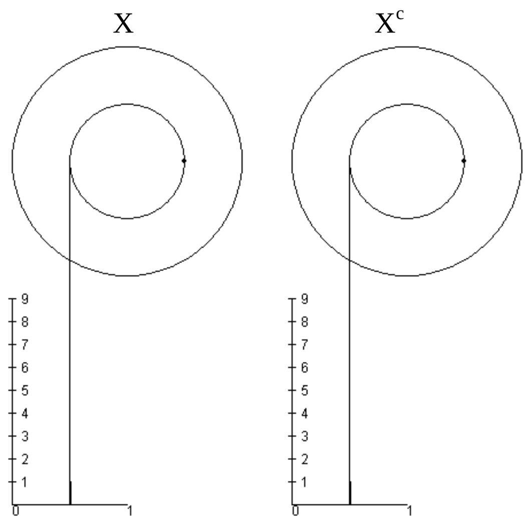

Figure 7. Visualization, According to the Existing Design, of a Fuzzy Set and its Complement Set when the Membership Function μ(x) = .5

The definition of the membership function of the complement of a fuzzy set on which our system is based, is given in Section 3.1. According to this definition the membership function and membership value for the complement of a fuzzy set are two different things. Therefore our system is able to produce the actual visual representation, shown in Figure 8, of a fuzzy set and its complement set when the membership function μ(x) = .5.

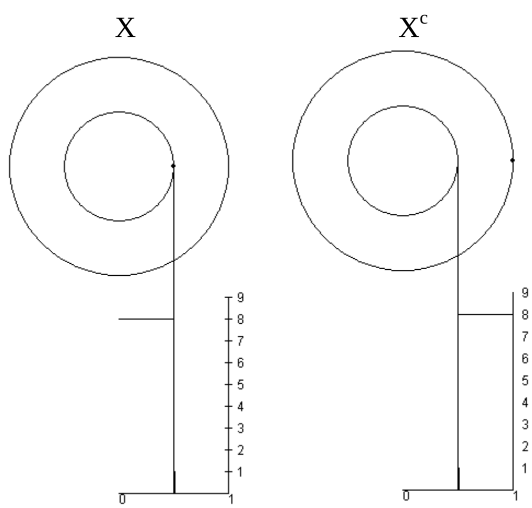

Figure 8. Visualization, According to our Design, of a Fuzzy Set and Its Complement Set when the Membership Function μ(x) = .5

3.1.2. Union of a Fuzzy Set with its Complement: In the existing design [2] the union of two fuzzy sets, A and B (say), has been defined as

$A \cup B = \{x, \max\{\mu_A(x), \mu_B(x)\}\}$, where $\mu_A(x)$ and $\mu_B(x)$ being the membership functions of A and B respectively.

Also in Section 3.1 it is discussed that according to Zadehian theory of fuzzy sets the membership function of complement A^c of a fuzzy set A is defined as 1- $\mu(x)$, where $\mu(x)$ is the membership function of A. Thus according to the existing design [2] which is based on Zadehian theory of fuzzy sets, the union of a fuzzy set A with its complement has been defined as

$$A \cup A^c = \{x, \max\{\mu_A(x), 1-\mu_A(x)\}\}$$

Now, with this definition and taking data from Table 2, the existing design produces the visual representation of $A \cup A^c$ as shown in Figure 9.

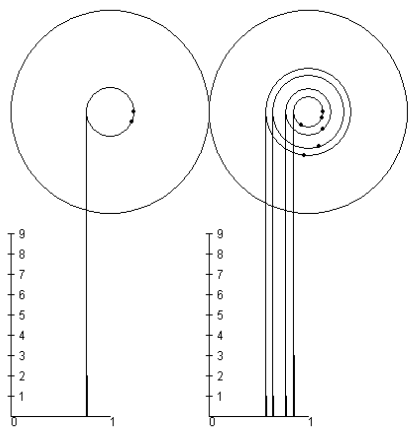

Figure 9. Visualization of Union of Fuzzy Set A with Its Complement According to the Existing Design

As Figure 9 does not represent the universal set, Ω, [3] so it can't be accepted as proper visualization of $A \cup A^c$. Through our design it is possible to produce the proper visualization $A \cup A^c$, i.e. $A \cup A^c = \Omega$. In our design we define [3]

$A \quad = \{x, \mu(x), 0; x \in \Omega\}$ and

$A^c \quad = \{x, 1, \mu(x); x \in \Omega\}$, where $\mu(x)$, 0 are the membership function and function of reference of x in fuzzy set A; 1, $\mu(x)$, are the membership function and function of reference of x in fuzzy set A^c and Ω is the universal set and their union is defined as

$$\begin{aligned} A \cup A^c &= \{x, \max\{\mu(x), 1\}, \min\{0, \mu(x)\}; x \in \Omega\} \\ &= \{x, 1, 0; x \in \Omega\} \\ &= \Omega \end{aligned}$$

Now, with this definition and taking the same data from Table 2 our design has produced the visual representation of $A \cup A^c$ as the universal set, Ω and which has been shown in Figure 10.

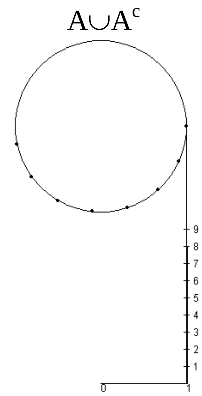

Figure 10. Visualization of Union of Fuzzy Set A with Its Complement According to our Design

3.1.3. Intersection of a Fuzzy Set with its Complement: In the existing design [2] the intersection of two fuzzy sets, A and B (say), has been defined as

$A \cap B = \{x, \min\{\mu_A(x), \mu_B(x)\}\}$, where $\mu_A(x)$ and $\mu_B(x)$ being the membership functions of A and B respectively.

Like union , similar argument can be put forward for intersection also i.e. according to the existing design [2] which is based on Zadehian theory of fuzzy sets the intersection of a fuzzy set A with its complement has been defined as

$A \cap A^c = \{x, \min\{\mu_A(x), 1-\mu_A(x)\}\}$

Now, with this definition and taking data from Table 2, the existing design produces the visual representation of $A \cap A^c$ as shown in Figure 11.

As Figure 11 does not represent the null set, ϕ, [3] so it can't be accepted as proper visualization of $A \cap A^c$. Through our design it is possible to produce the proper visualization $A \cap A^c$, i.e. $A \cap A^c = \phi$. In our design we define [3]

$A = \{x, \mu(x), 0; x \in \Omega\}$ and

$A^c = \{x, 1, \mu(x); x \in \Omega\}$, where $\mu(x)$, 0 are the membership function and function of reference of x in fuzzy set A, 1, $\mu(x)$, are the membership function and function of reference of x in fuzzy set A^c and Ω is the universal set and their intersection is defined as

$$\begin{aligned} A \cap A^c &= \{x, \min\{\mu(x), 1\}, \max\{0, \mu(x)\}; x \in \Omega\} \\ &= \{x, \mu(x), \mu(x); x \in \Omega\} \\ &= \phi \end{aligned}$$

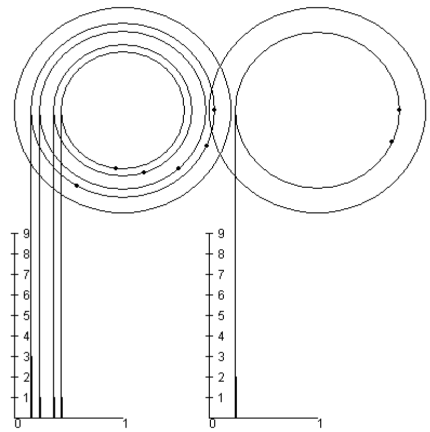

Figure 11. Visualization of Intersection of Fuzzy Set A with its Complement According to the Existing Design

Now, with this definition and taking the same data from Table 2, our design has produced the visual representation of $A \cap A^c$ as the null set, ϕ and which has been shown in Figure 12.

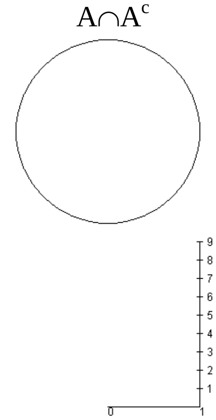

Figure 12. Visualization of Intersection of Fuzzy Set A with Its Complement According to our Design

4. Justification

Data elements in fuzzy sets are associated with membership functions between 0 and 1. The concentric circles in our design with maximum and minimum radii of 1 and 0 respectively, show the distribution of the elements within the fuzzy set with respect to the membership function and function of reference. From the circumferences of the circles one can easily find out respectively the membership function and function of reference of a fuzzy element in a fuzzy set. Although in general, the function of reference is not visible as it is considered to be zero for simplicity, its existence is quite visible especially in the case for the complement of a fuzzy set. One can easily find out the membership function, function of reference and membership value of a fuzzy element in a fuzzy set by observing the circumferences of the circles associated with the fuzzy element and the difference in their radii respectively. Also by observing the vertical bars which are the tangents to the circles in our design, one can easily find out the frequency of fuzzy elements with same membership function and function of reference.

5. Application

With this disk diagram for visualization of fuzzy set operations, it is possible to analyze fuzzy data and grasp the relationship among the sets which take part in our decision making. As for example in a word set, the distribution of words belonging to a particular topic set with respect to their membership function and function of reference can be visualized by our proposed design. This provides a user an insight how much the topic is associated with the word data. The user can also grasp the grouping of the words that belong to only one topic or to both in a word set by applying the standard fuzzy set operations of our design. The standard fuzzy set operations of our design can also be applied for web search engine that gives the relationship of web documents with membership function to users' query.

6. Implementation

Our design which does not follow the traditional Zadehian theory of fuzzy sets for the complement of a fuzzy set, but is based on the fact that the fuzzy membership value and fuzzy membership function for the complement of a fuzzy set are two different things [3], has been implemented using JAVA applet programming.

7. Conclusion

For a usual fuzzy set the fuzzy membership value and fuzzy membership function are not different because the value of the function is counted from zero in the usual case. Therefore we do not find any difference in visualization of usual fuzzy set with the existing design [2] which is based on the traditional Zadehian theory of fuzzy sets, and that with our design. But due to the absence of reference function in the existing deign, it could not visualize the complement of a fuzzy set properly (as discussed in section 3.1 & 3.1.1). Also, the existing design failed to visualize $A \cup A^c$ as the universal set, Ω and $A \cap A^c$ as the null set, ϕ (as discussed in section 3.1.2 and sec 3.1.3 respectively). Our design which is based on the fact that the fuzzy membership value and fuzzy membership function are two different things for the complement of a fuzzy set, has given the proper visual representation of the complement of a fuzzy set (as discussed in Section 3.1 & 3.1.1). Our design has also been able to visualize $A \cup A^c$ as the universal set, Ω and $A \cap A^c$ as the null set, ϕ (as discussed in section 3.1.2 and sec 3.1.3 respectively). Hence, we may conclude that the proper visualization of the elements in fuzzy set and standard fuzzy set operations like union, intersection and especially complement of a fuzzy set is possible only if we consider the fuzzy membership value and fuzzy membership function as two different things for the complement of a fuzzy set.

References

[1] B. Pham and R. Brown, "Visualization of Fuzzy Systems: Requirements, Techniques and Framework", Future Generation Computer Systems, vol. 21, no. 7, **(2005)**, pp. 1199-1212.

[2] Y. Park and J. Park, "Interactive Visualization of Fuzzy Set Operations", SPIE-IS&T, vol. 7530 753002, **(2010)**, pp. 1-12.

[3] H. K. Baruah, "Towards Forming a Field of Fuzzy Sets", International Journal of Energy, Information and Communications, vol. 2, Issue 1, **(2011)**, pp. 16-20.

[4] H. K. Baruah, "The Theory of Fuzzy Sets: Beliefs and Realities", International Journal of Energy, Information and Communications, vol. 2, Issue 2, **(2011)**, pp. 1 – 22.

[5] Q. S. Gao, X. Y. Gao and Y. Hu, "A New Fuzzy Set Theory Satisfying all Classical Set Formulas", Journal of Computer Science and Technology, vol. 24, no. 4, **(2009)**, pp. 798-804.