# On Geometrical Representation of Fuzzy Numbers

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## *Abstract*

*In this article, we intend to draw attention on the geometrical representation of the four fuzzy sets A, $A^c$ $A \cup A^c$ and $A \cap A^c$ and we would like to analyze it with the definition of complementation of fuzzy sets on the basis of reference function. Here efforts have been made to show that this kind of representation is not logical from the standpoints of new definition of complementation of fuzzy sets.It seems that the geometrical representations are the results of insufficient information and hence in order to meet the problem we would like to suggest that the complementation of fuzzy sets should be defined in the manner discussed in this article so that it becomes free from any controversy.*

*Keywords: fuzzy membership value, fuzzy membership function, Subsethood theorem, Entropy-Subsethood theorem*

## 1. Introduction

Fuzzy Set Theory was formalised by Professor Lofti Zadeh at the University of California in 1965. What Zadeh proposed is very much a paradigm shift that first gained acceptance in the Far East and its successful application has ensured its adoption around the world. In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition — an element either belongs or does not belong to the set. By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in a set , this is described with the aid of a membership function valued in the real unit interval [0, 1]. Fuzzy sets generalize classical sets, since the indicator functions of classical sets are special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1.Zadeh's fuzzy set theory may be described as follows:

Assume X is a classical set called Universe whose generic elements are denoted as x. A fuzzy subset A is denoted as $\{(x, \mu_A(x)): x \in X\}$ where $\mu_A(x)$ is the grade of membership of x in A. $\mu_A(x)$ is a real number satisfying $0 \leq \mu_A(x) \leq 1$ i.e $\mu_A(x) \in [0,1]$ where [0, 1] is a closed real interval.The complement of the fuzzy set A is denoted by $A^c$ and is defined by a membership function

$$\mu_{A^c}(x) = 1 - \mu_A(x), \forall x \in X$$

The concept of fuzzy set with varying degree of membership differs sharply from traditional mathematical theory because it violates laws of logic that date back to Greek philosopher Aristotle. In particular, fuzzy logic breaks both the law of contradiction and the law of excluded middle. Excluded middle laws are very important since they are the only set of operations that are not valid for both classical and fuzzy sets.

On the basis of this assumption, Kosko [1] had derived a proposition in which it was mentioned that a set is properly fuzzy if and only if $A \cap A^c \neq$ null set φ and if and only if $A \cup A^c \neq$ universal set Ω. In other words, he was in the view that fuzziness occurs only

when the excluded middle laws are violated Taking into consideration of this fact, the geometrical representation of fuzzy sets were introduced and on the basis of this, four fuzzy sets involved in the fuzziness of the set A- the sets A, $A^c$, $A \cup A^c$ and $A \cap A^c$ were represented graphically on the name of completing fuzzy square and it was also mentioned that these four fuzzy sets contract to the midpoint as A becomes maximally fuzzy. The geometry of fuzzy sets considered as a great aid in understanding fuzziness, defining fuzzy concepts and proving fuzzy theorems. Some fundamental questions of fuzzy set theory- How fuzzy are a fuzzy set? How much is one fuzzy set a subset of another?-were answered geometrically with Fuzzy Entropy Theorem, the fuzzy Subsethood theorem and Entropy – Subsethood Theorem. In other words, it can be said that this geometrical representation has become the cornerstone of many fuzzy theories. Additionally, it yielded derivation of the fuzzy set rules of intersection, union, and complement that previously been proposed by Zadeh.

Here in this article our main purpose is to highlight the shortcomings that exist in the definition of fuzzy sets and thereby to establish that those results which particularly involve complementation are not acceptable. There are many researchers in the field of fuzzy set theory who were also not satisfied with the way fuzzy sets are defined but our basis aim is to deal with the definition of complementation only because it causes confusion in logic and thinking.

## 2. Some Other Papers About Fuzzy Sets

M. Shimoda [11] presented a new and natural interpretation of fuzzy sets and fuzzy relations, but still did not change the fact that it could not satisfy all formulas of the classical set system.

A. Piegat [12] presented a new definition of the fuzzy set: a fuzzy set *A* of the elements *x* is a collection of the elements $\{x | x \in X\}$, which possess a specific property *pA* of the set and are qualified in the set by a qualifier *QA* using a qualification algorithm *QAlgA*. But nothing about essential shortcomings and mistakes of Zadeh's fuzzy set theory and how to overcome them completely was discussed in it.

Qing-Shi Gao, Xiao-Yu Gao and Yue Hu [13]found that there is some mistakes Zadeh's fuzzy sets and found that it is incorrect to define the set complement as $\mu_A c(\mathbf{x}) = 1 - \mu_A(\mathbf{x})$, because it can be proved that set complement may not exist in Zadeh's fuzzy set theory. According to them it leads to logical confusion, and is seriously mistaken to believe that logics of fuzzy sets necessarily go against classical and normal thinking, logic, and concepts.Since they found some shortcomings in the Zadeh's fuzzy set theory ,they wanted to move away from it and worked towards removing the shortcomings which according to them debarred fuzzy sets to satisfy all the properties of classical sets.They introduced a new fuzzy set theory,called *C*-fuzzy set theory which satisfies all the formulas of the classical set theory. The *C*-fuzzy set theory proposed by them was shown to overcome all of the errors and shortcomings, and more reasonably reflects fuzzy phenomenon in the natural world. It satisfies all relations, formulas, and operations of the classical set theory.

That is to say that these authors are also not satisfied with the way of defining the complemention of fuzzy sets.There are many such cases where the researchers found some sort of problems in the Zadeh's fuzzy set theory some of which are mentioned above. In this article, we would like to rivisit the definition of complementation of fuzzy sets and in due course would like to relace it with a new one so that it becomes free from any doubt. But before proceeding further, let us have a brief view of the geometrical representation of fuzzy sets.

## 3. Geometrical Representations of Fuzzy Sets

Bart Kosko introduced a very useful graphical representation of fuzzy sets. Figure 1 shows an example in which the universal set consists only of two elements $x_1$ and $x_2$. Each point of the interior of the unit square represents a subset of X. Here, it was assumed that the co-ordinates of the representation correspond to the membership values of the elements in the fuzzy set. The Universal set X is represented by the point (1, 1) with membership function

$$\mu_A(x_1) = 1 \text{ and } \mu_A(x_2) = 1.$$

The point (1, 0) represents the set $\{x_1\}$ and the point (0, 1) represents the set $\{x_2\}$. The crisp subsets of x are located at the vertices of the unit square. In geometrical representation, the point (0, 0), for example, is interpreted as $x_1/0 + x_2/0$ implying that neither $x_1$ nor $x_2$ is a member of the universal set X and similarly for others.

Let us have a look at the proposed geometrical representation of the four fuzzy sets A, $A^c$ $A \cup A^c$ and $A \cap A^c$ with the help of the following diagram.
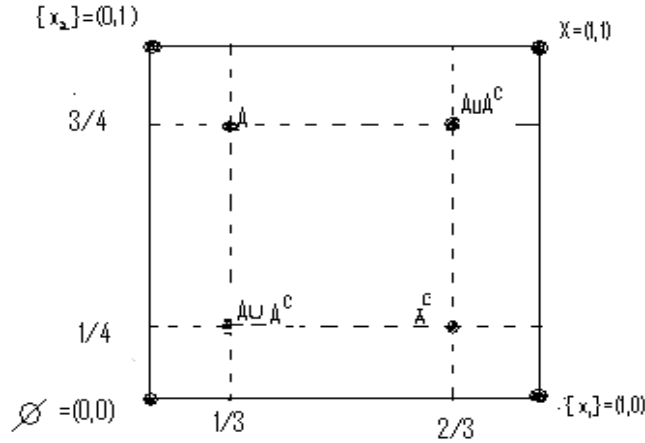


**Figure 1. Geometrical Representation of Fuzzy Sets**

In this article, our purpose is to show that this kind of graphical representation is not logical if it is seen from the standpoints of the new definition of complementation and hence it cannot be considered as a strong base for answering the various fundamental questions that would arise while dealing with fuzzy set theory. Here, efforts have been made to show that the aforesaid proposition as well as the representation of the four fuzzy sets becomes illogical. It is to be pointed out here that the laws which had been there since the Zadeh's initial conception are unacceptable but whatsoever these cannot be rejected without having a proper mathematical tool. For this purpose the, the most appropriate way would be to define complementation of fuzzy sets on the basis of reference function as defined by Baruah [2]. This activity is more mathematical or formal in character and hence it can be of great help in establishing our claim. Hopefully this may be helpful in removing the shortcomings which are there in the existing definition of fuzzy complementation. The procedure can now be described as follows.

## 4. Baruah's Definition of Complementation of Fuzzy Sets

It is observed that in Zadeh's definition of complementation no distinction is made between fuzzy membership function and fuzzy membership value. According to Baruah [2, 3 & 4], it is necessary to define a fuzzy set with the help of two functions namely: membership function and reference function. Fuzzy membership value is taken to be the difference between fuzzy membership function and fuzzy reference function. Accordingly, a fuzzy number N is defined with the help of a fuzzy membership function $\mu_2(x)$ (say) and a reference function $\mu_1(x)$ (say) such that both lie between 0 and 1 under the condition $\mu_2(x) \geq \mu_1(x)$ .Then for a fuzzy number denoted by $\{x, \mu_1(x), \mu_2(x), x\epsilon\Omega\}$ , the fuzzy membership value will be defined by $\{\mu_2(x) - \mu_1(x)\}$ in the proposed manner and this is different from fuzzy membership function. In the process of defining complementation in the aforesaid manner, a difference is maintained between fuzzy membership value and fuzzy membership function.

Again it is worth mentioning here that the existence of two laws randomness is required to define a law of fuzziness Baruah [4]. In otherwords, it was found that two distributions with reference to two laws of randomness defined on two disjoint spaces can construct a fuzzy membership function. That is to say, a proper way is to bifurcate the nembership function into two parts and it is necessary to introduce two probability spaces, one to define distribution function and the other to define complementary distribution function.

Accordingly, the membership function $\mu_N(x)$, of a normal fuzzy number N= $[\alpha, \beta, \gamma]$ is defined in the following way:

$$\mu_N(x) = \psi_1(x), \text{ if } \alpha \leq x \leq \beta$$

$$= \psi_2(x) , \text{ if } \beta \leq x \leq \gamma$$

$$= 0, \text{ otherwise.}$$

$$\text{while } \psi_1(\alpha) = \psi_2(\gamma) = 0 , \text{ and}$$

$$\psi_1(\beta) = \psi_2(\beta) = 1$$

Here $\psi_1(x)$ and $\psi_2(x)$ stand for distribution function and complementary distribution function respectively. Further, it is to be noted that $\psi_1(x)$ is continuous and non-decreasing function in the interval $[\alpha, \beta]$ and $\psi_2(x)$ is continuous and non-increasing function in the interval $[\beta, \gamma]$

As a consequence of the above result, it can be seen that the complement $N^c$ of the normal fuzzy number N will be equal to 1 for the entire real line; the membership value would have to be counted from the membership value of N.

In accordance with the process discussed above, a fuzzy set defined by

$A = \{x, \mu(x), x\epsilon\Omega\}$

would be defined in this way as

$A = \{x, \mu(x), 0, x\epsilon\Omega\}$

so that the complement would become

$A^c = \{x, 1, \mu(x), x\epsilon\Omega\}$

The extended definition using a reference function leads to the assertion that for any fuzzy set A we have

$$A \cap A^c = \text{the null set } \varphi \text{ and}$$

$$A \cup A^c = \text{the universal set } \Omega$$

Here we see that the two most debatable properties of fuzzy sets will not remain debatable any more in the view of extended definition of complementation of fuzzy sets. In other words the two laws which were assumed to be true only for classical sets hold for fuzzy sets also. Thus it is observed that if the complementation of fuzzy sets is defined in the manner stated above then the proposition as well as the geometrical representation becomes unworkable from the new perspective. In our case $A \cup A^c$ represents the whole universe and hence it coincides with the point X in the fuzzy square and $A \cap A^c$ is the null set which represents the point $\varphi$ that is (0,0) in the in the so called fuzzy square and hence we can claim that such types of representation which are based on the assumption that the complementation of a fuzzy set can only be defined as one minus the membership function of the set is not a valid representation .Again it is to be noted that the geometry of fuzzy sets cannot be of great help in understanding fuzziness and proving fuzzy theorems because the representations which include union and intersection of a fuzzy set with its complement cannot give an authentic representation if we proceed in our way.Moreover, it causes a mistaken belief that the logic of fuzzy sets would necessarily go against classical and normal thinking ,logic and concepts.From the above discussions, it is clear that when we consider the case of complementation then the representation seems to have no such forms as shown in the figure. Hence we would like to say that if the complement cannot be represented in that proposed manner then there will be no use of such graphical representations. So it can be mentioned here that the Fuzzy Entropy Theorem, the fuzzy Subsethood theorem and Entropy – Subsethood Theorem which were found on the basis of this geometry have nothing to do from our standpoints.That is to say that the geometrical representation of the aforesaid theorems thus become inappropiate and hence proper care should be taken before using the results in further works.

## 5. Conclusions

In this short work efforts are made to show that proposed representation of four fuzzy numbers in the said manner is not at all logical in our standpoint and hence becomes unacceptable. Again the proposition in which it was mentioned that a set is fuzzy if and only if the excluded middle laws are violated , can no longer be taken for granted if we look into it, keeping in mind the definition of complementation of fuzzy sets in the manner suggested in this article. Here in this article we have noticed that neither the proposition nor the graphical representation of the four fuzzy sets as stated earlier has some logical bearing. Hence it can be said that these results cannot be used in other fuzzy theorems which were mentioned earlier. So this matter should be taken care of for future works.

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