A Note on the Existing Definition of Fuzzy Entropy

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Abstract

In this note, we shall deal with entropy of fuzzy sets. Finding entropy of a fuzzy set is very important in the application of fuzzy set theory. Until now, there have been several methods to measure the fuzziness of a fuzzy set. The objective of this article is to show how such definitions fails to give a reliable measure of entropy of fuzzy sets, if we use an extended definition of complementation of fuzzy sets using reference function, which in turn prompts us to define fuzzy entropy in a different way with reference to Shannon's entropy.

Keywords: *Fuzzy membership value, fuzzy membership function, cardinality, fuzziness.*

1. Introduction

There often exists uncertainty in real systems. The entropy of fuzzy sets is a measure of fuzziness between fuzzy sets. A number of approaches to this end have become known. Different authors have defined it in the way which seemed to them to be appropriate. In this article, we would like to question the property of the one which was proposed by Rogas [1]. Rogas[1] has defined the entropy of a fuzzy set A as:

$$E(A)=\frac{|A\cap A^c|}{|A\cup A^c|} \qquad (1)$$

where$|A \cap A^c|$and $|A \cup A^c|$denote the cardinalities of the sets $A \cap A^c$ and $A\cup A^c$, where A^c stands for the complement of the set A which is defined with the help of the membership function

$$\mu_A c\ (x) = 1 - \mu_A(x), \forall x\epsilon\ \Omega$$

While defining entropy in this manner, it was assumed that the two laws –the Law of Non-contradiction and the Law of Excluded Middle are violated by fuzzy sets.
In this article, we shall show how this definition of entropy will always give us the result zero even inthe case of fuzzy set with the help of definition of complementation of fuzzy sets based on reference function as proposed by Baruah [2] on the basis of superimposition of sets.

2. Baruah's Definition of Extended Fuzzy Sets

Baruah [2, 3] has defined a fuzzy number N with the help of two functions :a fuzzy membership function $\mu_2(x)$ and a reference function $\mu_1(x)$ such that $0 \leq \mu_1(x) \leq \mu_2(x) \leq 1$ Then for a fuzzy number denoted by $\{x, \mu_1(x), \mu_2(x), x\epsilon\Omega\}$ we would call $\{\mu_2(x) - \mu_1(x)\}$

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as the fuzzy membership value ,which is different from fuzzy membership function .It is to be noted here that in the definition of complement of a fuzzy set, fuzzy membership value and the fuzzy membership function have to be different, in the sense that for a usual fuzzy set the membership value and membership function are of course equivalent. This can be visualized with the help of following diagram:

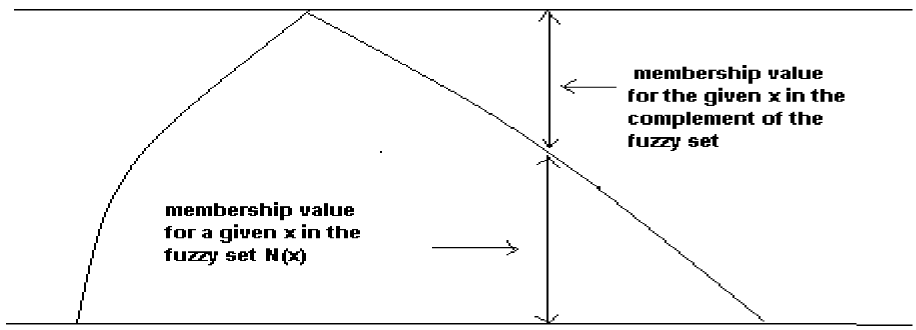

Figure 1. Baruah's Definition of Extended Fuzzy Sets

For a normal fuzzy number N= $[\alpha, \beta, \gamma]$ defined with a membership function $\mu_N(x)$, where

$$\mu_N(x) = \psi_1(x), \text{ if } \alpha \le x \le \beta$$
$$= \psi_2(x) \text{ , if } \beta \le x \le \gamma$$
$$= 0, \text{ otherwise.}$$

while $\psi_1(\alpha) = \psi_2(\gamma) = 0$, $\psi_1(\beta) = \psi_2(\beta) = 1$,the complement N^c will have the membership function

$$\mu_N^c(x) = 1, -\infty < x < \infty$$

with the condition that $\mu_N^c(x)$is to be counted from $\psi_1(x)$, if $\alpha \le x \le \beta$,from $\psi_2(x)$, if $\beta \le x \le \gamma$ and from zero otherwise. Accordingly, we have defined the fuzzy membership function of the complement of a normal fuzzy number N to be equal to 1 for the entire real line, with membership value counted from the membership value of N. The extended definition using a reference function leads to the assertion that for any fuzzy set A we have

$$A \cap A^c = \text{the null set } \varphi \text{ and}$$

$$A \cup A^c = \text{ the universal set } \Omega$$

In other words the two laws which were assumed to be true only for classical sets hold for fuzzy sets also.

3. Entropy of a Fuzzy Set

If we apply the result obtained above then we have $A \cap A^c =$ the null set φ and hence the cardinality of this set is always zero according to the definition of cardinality. Now if the cardinality of the fuzzy set is zero then the entropy defined in the manner (1)will reduce to $E(A) = 0$ always. We will never be able to find any measure of fuzziness from this proposed definition. Again, since it is said that the entropy of a crisp set is zero, so here we cannot get the entropy of a fuzzy set with the help of our definition of complementation of normal fuzzy sets.

Keeping in view of the original definition of Shannon's entropy which is defined as:

$$H=-\sum p_i log p_i \quad ,i=1,2,\ldots,n.$$

where(p_1, $p_{2,\ldots\ldots,}p_n$) denotes the probabilities of n events, fuzzy entropy too can be defined by using the Randomness-Fuzziness Consistency Principle defined by Baruah [3] as:

For a normal fuzzy number N= $[\alpha, \beta, \gamma]$ defined with a membership function$\mu_N(x)$, where

$$\mu_N(x) = \psi_1(x), \text{ if } \alpha \leq x \leq \beta$$

$$=\psi_2(x) \text{ , if } \beta \leq x \leq \gamma$$

$$= 0, \text{ otherwise.}$$

With$\psi_1(\alpha) = \psi_2(\gamma) = 0$,$\psi_1(\beta) = \psi_2(\beta) = 1$, the partial presence of a value x of a variable X in the interval $[\alpha, \gamma]$is expressible as:

$$\mu_N(x) = \theta \text{ Prob } [\alpha \leq X \leq x] + (1- \theta) \{1\text{-Prob}[\beta \leq X \leq x] \}$$

where if $\theta = 1, \alpha \leq x \leq \beta,$ if $\theta = 0,\ \beta \leq x \leq \gamma$. So, $\mu_N(x)$ for $\alpha \leq x \leq \gamma$ is nothing but a probability density only, either as $Prob[\alpha \leq X \leq x]$ or as $\{1 - Prob[\beta \leq X \leq x]\}$ whichever is the case.This definition takes into account the fact that the membership function explaining a fuzzy variable taking a particular value is either the distribution function of a random event or the complementary distribution function of another random event. It was thus established that two laws of randomness are needed to define one possibility law.

Accordingly, the left reference function of a normal fuzzy number which is nothing but a distribution function, would lead to an entropy E_1.In a similar manner, the right reference function of the normal fuzzy number, which is nothing but a complementary distribution function, would lead to another entropyE_2.The pair $[E_1 , E_2]$ found can rightly be called fuzzy entropy in the classical sense of defining Shannon's entropy for a discrete law of randomness. Discretizing a law of randomness for a continuous variable should not be of much problem, which in turn can be used to define fuzzy entropy $[E_1 , E_2]$, where E_1 and E_2 are Shannon's entropies for the left reference function and right reference function respectively.

4. Numerical Example

Let X= [4, 16, 25] be a fuzzy number with membership function defined as:

$$\mu_X(x) = \frac{\sqrt{x} - 2}{2}, 4 \leq x \leq 16,$$

$$= 5 - \sqrt{x} \text{ },16 \leq x \leq 25,$$

$$= 0, otherwise$$

Here we shall find Shannon's entropy for the left reference function and the right reference function respectively for the above mentioned fuzzy number with given fmf, with the help of which our proposed definition of entropy would be more clear.

Table 1. Shannon's Entropy for the Left Reference Function

x	F(x)	p	lnp	plnp
4.0	0	.2649111	-1.3283610	-0.3518957
6.4	.2649111	.2183286	-1.5217540	-0.3322424
8.8	.4832397	.1900804	-1.6603081	-0.3155920
11.2	.6733201	.1705888	-1.7684993	-0.3016862
13.6	.8439089	.1560911	-1.8573155	-0.2899104
16.0	1			1.5913267

Table2: Shannon's Entropy for the Right Reference Function

x	G(x)	1-G(x)	p	lnp	plnp
16.0	1	0	.2190046	-1.5186625	-0.3325940
17.8	.7809954	.2190046	.2081841	-1.5693325	-0.3267100
19.6	.5728113	.4271887	.1988247	-1.6153317	-0.3211678
21.4	.3739866	.6260134	.1906244	-1.6574503	-0.3159504
23.2	.1833622	.8166378	.1833622	-1.69629184	-0.3110358
25.0	0	1			1.607459

Thus with five equal intervals, the discreatize points of x for the left and right reference function are:

A={(4,0),(6.4,.26),(8.8,.48),(11.2,.67),(13.6,.84),(16,1),(17.8,.78),(19.6,.57),(21.4,.37),(23.2,.18), (25.0,0)}.

The pair of Shannon's entropy here found to be (1.591327, 1.607459)

Let us see what it would have been, if proceeded with the formula mentioned in (1) .Here the complement was defined as one minus the membership function and accordingly we shall get

$A^c = \{$ (4,1),(6.4,.74),(8.8,.52),(11.2,.33),(13.6,.16),(16.0),(17.8,.22),(19.6,.43),(21.4,.63),(23.2,.82), (25.0, 1)}

$A \cup A^c = \{(4, 1)$,(6.4,.74),(8.8,.52),(11.2,.67),(13.6,.84),(16,1),(17.8,.78),(19.6,.57),(21.4,.63),(23.2,.82), (25,1)}

$A \cap A^c$ ={(4,0),(6.4,.26),(8.8,.48),(11.2,.33),(13.6,.16),(16,0),(17.8,.22),(19.6,.43),(21.4,.37),(23.2,.18), (25,0)}

Hence E(A) = $\frac{2.43}{8.57} = .283547$

Thus with the above example, we tried to find entropy of a fuzzy set in both the way we mentioned here and from these it is expected that our proposed method would be clear.

5. Conclusions

In this note, an attempt is made to show how the existing definition is unable to provide us are reliable method to measure the fuzziness of a fuzzy set. Here we have first gone through the definition of entropy of fuzzy sets given by Rogas and then proposed a new way of defining it. Again, in this article, we have shown how existing definition of fuzzy entropy always yields the result zero if the extended definition of complementation of fuzzy set using reference function is taken into consideration. We have made an attempt is made to provide a better and reliable way to consider fuzzy entropy of a normal fuzzy number with reference to Shannon's entropy.

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