Non-Probabilistic Uncertainty Analysis of Analytical and Numerical Solution of Heat Conduction

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Abstract

Fuzzy set theory is applied to quantify the non-probabilistic uncertainty alternatively termed as epistemic uncertainty. An algorithm "Fuzzy Centered Radius" has been developed for quantification of epistemic uncertainty. Uncertainty analysis is also carried out using fuzzy vertex method. Analytical solution of one dimension heat conduction and numerical solution of heat removal from circular fin are applied to quantify the uncertainty of the temperature distribution using the proposed algorithm. Heat transfer coefficient and thermal conductivity are considered as fuzzy parameter due to their imprecise measured values. Results of uncertainty of temperature distribution based on FCR algorithm in case of analytical solution of heat conduction is compared with that obtained on the basis of fuzzy vertex method. Finite element method with the specified fuzzy parameters is adopted to obtain the membership values of the temperature at the specified node and the heat removal rate from the fin. Uncertainty in both the cases is estimated in terms of fuzziness measure.

Keywords: *uncertainty, fuzzy vertex theory, heat transfer, thermometric conductivity*

1. Introduction

Uncertainty can arise from different sources (complexity, ignorance, randomness, imprecision, vagueness, etc) [1]. Concerning to most of the real problems, two types of uncertainty exists. Uncertainty based on the randomness (variability) of the model parameter is termed as aleatory uncertainty whereas, uncertainty based on fuzziness (uncertainty) of the model parameter is termed as epistemic uncertainty. Thus variability and imprecision are two distinct facets of uncertainty analysis. Variability is alternatively called as "objective uncertainty" that arises from heterogeneity or the random character of natural processes. Imprecision is referred to as "subjective uncertainty" arises from the partial character of our knowledge of the natural world. The randomness type of uncertainty has long been handled appropriately by probability theory and classical statistics. In fact, prior to the entry of fuzzy set theory into the mathematical repertory, the only well developed mathematical apparatus for dealing with uncertainty was probability theory, that is to say that probability theory is appropriate for dealing with aleatory uncertainty. Hence, to quantify the vagueness (fuzziness) uncertainty (epistemic uncertainty) we need a new mathematical framework. Fuzzy sets provide a mathematical tool to deal with such uncertainty in realistic systems. In sort, fuzzy set theory is related to the non-precise data, vague statements, imprecise environments and fuzzy structures. The application of fuzzy set theory [2] proves to be a practical approach for solving the limitation of handling the uncertainties in the model parameters that can be taken into account by representing the effects of scatter by fuzzy numbers with their shape derived from statistical data.

Here, in this paper, the issue of uncertainty quantification [2] in the analytical and numerical solution of heat conduction problem is focused using a new fuzzy centered radius (FCR) algorithm. Fuzzy vertex method, introduced by Dong and Shah(1987) is applied for uncertainty quantification to verify the results obtained using FCR algorithm [3]. Analytical solution and numerical solution of heat conduction problem are basically considered for uncertainty analysis. Analytical solution of one dimension transient diffusion equation representing the heat conduction problem is considered for uncertainty analysis of the temperature of the system under consideration. Standard circular fin problem is considered for uncertainty analysis of numerical solution of the temperature distribution and heat removal rate. The coefficients of the heat conduction equation possess the material properties such as thermal conductivity, density, specific heat and heat transfer coefficient. These property values are imprecise due to insufficient measurements. Accordingly, these properties can be addressed as fuzzy. In this paper, thermal conductivity and heat transfer coefficient are considered as fuzzy parameters, whereas, density and specific heat of the material are taken as deterministic parameters. The boundary conditions required for solving the heat conduction problem may also be fuzzy due to imprecise specification of parameter of interest. Therefore, solution of the partial differential equation representing the heat conduction problem of interest may not be crisp, rather it will be a fuzzy number. Membership values of the imprecise or fuzzy parameters can be generated on the basis of their alpha level representation. Two case studies are presented in the paper. Case study (1) describes the uncertainty of the temperature for the flowing of heat through a circular bar and case study (2) presents the uncertainty of temperature during the dissipation of heat through a circular fin. One dimension heat transfer equation is used in both the cases. Estimation of uncertainty of the analytical and numerical solution of temperature for both the cases is addressed using fuzziness measure of uncertainty. The paper is organized in the following way: Section 2 describes in brief various mathematical approaches of handling uncertainties. Section 3 presents the fuzzy set theoretic approach (Fuzzy vertex method and fuzzy centered radius method) of handling epistemic uncertainty. Section 4 presents the results of the case studied as an implementation of non-probabilistic method of uncertainty analysis. Section 5 gives the conclusions.

2. Literature Review of Methodologies of Uncertainty Analysis

A large number of mathematical methods have been developed for handling the uncertainties. Probabilistic and statistical methods are usually preferred for analysis of aleatory uncertainty. On the other hand, several competing approaches have been suggested when both aleatory and epistemic uncertainties present. Aleatory uncertainty is irreducible due to the variability of the system; however, epistemic uncertainty is reducible in the sense of updating knowledge by gathering more information related to the problem. The most important methods of handling uncertainty consist of interval analysis [4, 5, 6], fuzzy set theory [7], possibility theory [7], probability bounds theory [8], and different hybrid methods [9]. The common feature of these methods is that instead of single valued crisp input data they apply different new types of data expressing the amount of uncertainty related to the given input data. Interval analysis [3, 4] replaces crisp numbers by uncertainty intervals (figure 1.0). Interval analysis lacks gradations and is the simplest method to express uncertainty through arithmetic calculations. The method guarantees that the true value will always remain within the interval, but this goal is achieved at cost of precision. During the calculations the intervals become wider and wider and the final results become too conservative.

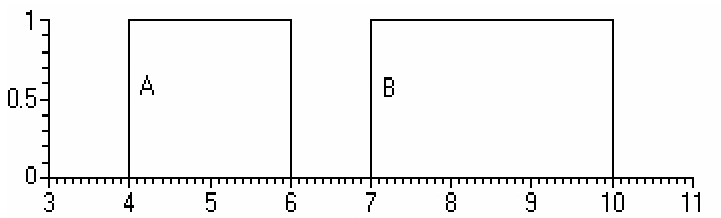

Figure 1. Representation of Crisp Numbers using Intervals

Possibility theory [7], a generalization of interval analysis, provides a suitable model for the quantification of uncertainty by means of the possibility of an event [7]. The membership value of a number, varying between zero and one, expresses the plausibility of the occurrence of that number. They represent estimates of uncertainty at different levels of possibility (or membership degree). The related fuzzy set theory expresses uncertainty very often by the use of fuzzy numbers. Membership functions of fuzzy numbers are by definition convex and the values reaching the possibility level one are considered as the most possible estimates of the given variable. This interval is called the core of the fuzzy number. Alpha level representation of two fuzzy sets with their triangular membership functions for fuzzy arithmetic operation is clarified in figure 2.0. It can be explained from Figure 2.0 that, α_4-level of fuzzy sets x_1 and x_2 is deterministic whereas α_1, α_2 and α_3 level of x_1 and x_2 are interval valued. The fuzzy output shown in Figure 2.0 is obtained by taking all possible combinations of alpha-levels of $x_1$and x_2.

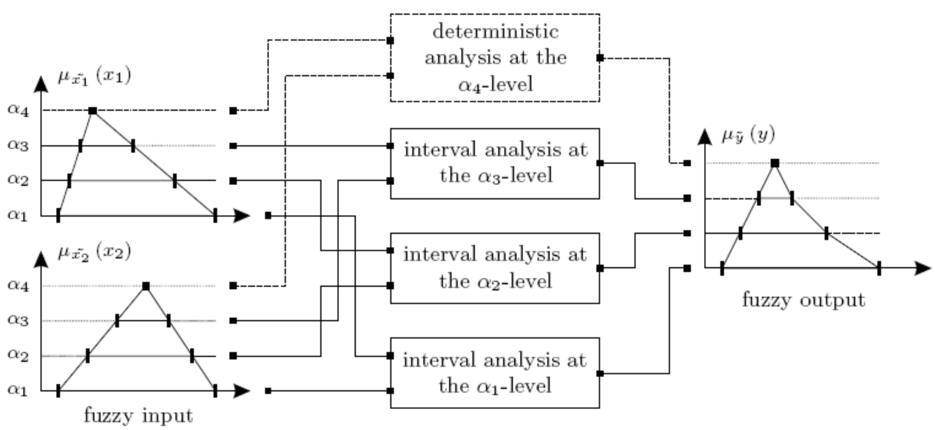

Figure 2. α-level Strategy with 4 α-levels, for a Function of Two Triangular Fuzzy Parameters

3. Uncertainty Analysis Using Non-Probabilistic Method

3.1 Role of Fuzzy Arithmetic

Mathematically the fuzzy set A is defined by using equation (1)

$$A = \{(x,\mu(x)) \mid (x \in X)\ (\mu(x) \in [0,1])\ \} \quad (1)$$

The shape of the membership function for the present usage in quantifying the epistemic uncertainty is considered as triangular, whose membership function is represented by the equation (2)

$$\mu(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \le x \le b \\ \dfrac{c-x}{c-b}, & b \le x \le c \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

In order to compute the uncertainty of the model of interest, fuzzy arithmetic has been used in which the arithmetic operation is performed using α cut of the specified fuzzy set $\{x, \mu_A(x)\}$. The basic fuzzy arithmetic operations of two fuzzy sets A and B used to compute the uncertainty of the model are given by equations (3-6).

1) Addition: $A_\alpha + B_\alpha = [a_1^\alpha, a_2^\alpha] + [b_1^\alpha, b_2^\alpha] = [a_1^\alpha + b_1^\alpha , a_2^\alpha + b_2^\alpha]$ (3)
2) Subtraction: $A_\alpha - B_\alpha = [a_1^\alpha, a_2^\alpha] - [b_1^\alpha, b_2^\alpha] = [a_1^\alpha - b_2^\alpha , a_2^\alpha - b_1^\alpha]$ (4)
3) Multiplication: $A_\alpha * B_\alpha = [a_1^\alpha, a_2^\alpha] * [b_1^\alpha, b_2^\alpha]$
$= [\min(a_1^\alpha b_1^\alpha, a_1^\alpha b_2^\alpha, a_2^\alpha b_1^\alpha, a_2^\alpha b_2^\alpha),$
$\max(a_1^\alpha b_1^\alpha, a_1^\alpha b_2^\alpha, a_2^\alpha b_1^\alpha, a_2^\alpha b_2^\alpha)]$ (5)
4) Division: $A_\alpha / B_\alpha = [a_1^\alpha, a_2^\alpha] * [1/b_2^\alpha, 1/b_1^\alpha]$ (6)
(Similar to multiplication rule (3)

Handling uncertainty for the solution of a differential equation using fuzzy arithmetic presents a new paradigm of numerical solution of differential equation with imprecise parameters as coefficients of the governing differential equation. The complete mathematical description of the new numerical method of fuzzy finite element (FFEM) is described in [10, 11, 12]. The details of the interval arithmetic are described elsewhere in [2, 4].

Uncertainty analysis using fuzzy arithmetic approach has been carried out using fuzzy vertex method [13]. Algorithm of fuzzy vertex method (FVM) used for the present work is as follows:

1. Represent input uncertain parameters as fuzzy number (here it is triangular fuzzy number).
2. Discretize the range of membership grade [0,1] into a finite number of α-cut levels.
3. Construct the representative intervals for each fuzzy set at the specified α-cut levels.
4. Take one end point from each of the intervals (under an α-cut level). Note that, there are 2^n combinations for n fuzzy sets.
5. Solve 2^n deterministic sub models for each lower and upper bound of the model output function.

Finally, solutions of the sub models are integrated by max-min rule and report the final solution for the output function at any α-cut level as [lower bound, upper bound].

3.2 Fuzzy Centered Radius Algorithm

Fuzzy Centered Radius (FCR) algorithm is based on alpha cut of a fuzzy set. Algorithm is proposed for reducing the interval between the upper and lower bounds of the final solution of the problem of interest. Algorithm first generates the alpha cut of fuzzy or imprecise parameters of the model on the basis of their specified membership function. Triangular membership function of the fuzzy parameters is used here for implementing the FCR algorithm. This is because the triangular membership function is the simplest and possesses the convex property. Each interval representing the specific alpha cut is modified as a paired number wherein first number is termed as centre of the interval and the second number is defined as the radius of the interval. Centre of the interval is computed as the average of the bounds of the interval and radius is computed as the half width of the interval. So, if [a,b] represents an interval at any alpha cut of a fuzzy set, then as per FCR algorithm centre, C = (a+b)/2 and radius, R = (b-a)/2. So, using FCR algorithm, paired number (C, R) for each alpha cut of a fuzzy number is generated. Finally fuzzy vertex method on these paired numbers (C, R) generated for fuzzy parameters of the specific model is applied to compute the bounds of the solution set for each alpha cut.

3.3 Non-probabilistic Uncertainty

Non-probabilistic uncertainty is alternatively called as epistemic uncertainty which is due to the subjective assignment of the objective consensus. Generally, quantification of epistemic uncertainty is expressed as the ratio of the interval of the bounds (upper – lower) at 0.1 alpha cut level to the value at alpha cut equal to 1. Probabilistic analogy of this quantity is coefficient of variation defined as the ratio of the standard deviation to the mean of the probabilistic output. However, in this paper, uncertainty is quantified in terms of fuzziness measure of uncertainty. The choice of fuzziness measure to quantify the epistemic uncertainty is due to the fact that fuzziness specifies how much fuzziness of the fuzzy output is, so that through the propagation of fuzziness, it can be proved that fuzziness exists at the beginning stage of a system, that is when the information or knowledge is insufficient. More the gain of the knowledge, completeness of the information results and subsequently fuzziness decreases.

Fuzziness of a fuzzy set A is defined as the sum of the lack of distinction of the set and its complement. The local distinction (one for each x $\in$ X) of a given fuzzy set, A and its complement is defined as Hamming distance and is measured by

$$\left|A(x)-(1-A(x))\right| = \left|2A(x)-1\right| \tag{7}$$

The lack of each local distinction is measured by

$$1-\left|2A(x)-1\right| \tag{8}$$

The measure of fuzziness, f(A) is then obtained by adding all these local measurements:

$$f(A) = \sum_{x \in X} \left(1-\left|2A(x)-1\right|\right) \tag{9}$$

For a symmetric triangular fuzzy number, A (a_L, a_m, a_R) with X = [a_L, a_R] measure of fuzziness, $f(A)$ using equation (9) can be written as

$$f(A) = \int_{a_L}^{a_m} \left(1-\left|2\frac{x-a_L}{a_m-a_L}-1\right|\right) dx + \int_{a_m}^{a_R} \left(1-\left|2\frac{x-a_R}{a_m-a_R}-1\right|\right) dx \tag{10}$$

4. Case Study

Two problems were addressed for uncertainty quantification using fuzzy set theory. First problem describes the uncertainty quantification of the temperature of a material obtained on the basis of the analytical solution of one dimension heat conduction equation. Second problem presents the uncertainty analysis of the temperature for the heat dissipation from a circular fin. This problem also presents the uncertainty analysis of heat removal rate from a circular fin. Finite element method with imprecise values of thermal conductivity and heat transfer coefficient is applied for obtaining the fuzzy numerical solution of the temperature of the circular fin.

4.1 Case 1: One Dimensional Transient Heat Conduction

In this case we have considered the flow of heat through a rectangular bar which in one dimension is represented as:

$$\frac{1}{\kappa}\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2} \tag{11}$$

where, the constant, κ (= K/ρC) signifies the thermometric conductivity, K represents the thermal conductivity, ρ is the density of the material and C is the specific heat of the material of the body. From the point of precision in measurements, measured values of these parameters are imprecise and hence one can assume the presence of uncertainty in K, ρ and C separately. Finally one can determine the uncertainty associated with the derived constant, κ. However, here, in this paper, only thermal conductivity is taken as fuzzy parameter and density and specific heat are taken as deterministic. So, uncertainty in κ is only due to the uncertainty of thermal conductivity, K. Uncertainty in κ is prescribed as a triangular fuzzy number specified by the membership function shown in figure 3.0. Initial and boundary conditions for this problem are prescribed as: T(x, 0) = 0 and T(0,t) = T_0. The temperature T_0 being measured possess an uncertainty which is further prescribes as fuzzy governed by the membership function depicted in figure 3.0. It represents the uncertainty in measurement of initial temperature. We are required to carry out an uncertainty analysis of the temperature distribution in the semi infinite region x > 0.

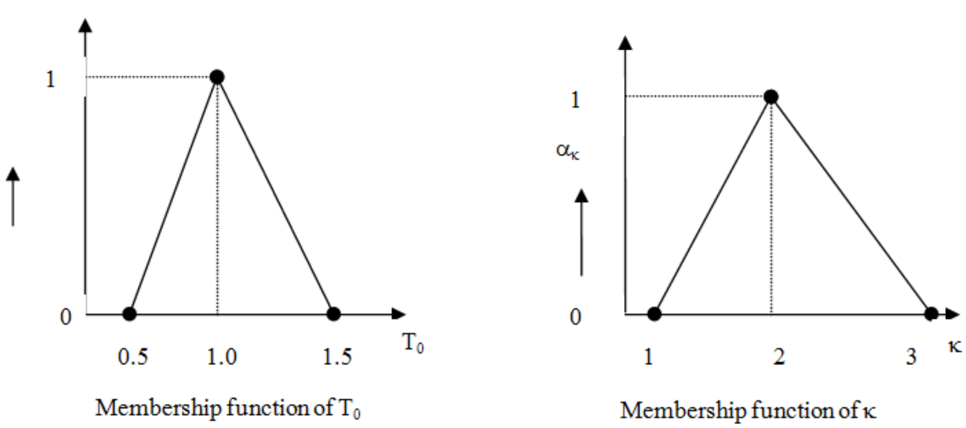

Figure 3. Membership Functions of Uncertain (Fuzzy) Parameters

Analytical solution of the problem (equation (11) can be written as

$$T(x,t) = T_0 \left[1 - erf\left(\frac{x}{2} \sqrt{\frac{1}{\kappa t}} \right) \right] \tag{12}$$

where, T_0 is the temperature at x = 0, κ, the thermometric conductivity (diffusivity), t is the time of observation and x is the spatial distance along the length of the bar from the origin of the source. The temperature, T_0 and the parameter, κ being fuzzy are expressed in their alpha –level representation, i.e, left-right bounds (L-R) form and all alpha cut values of T_0 and κ using that representation can be written as

$$(T_0)_\alpha^L = 0.5 + \alpha_T\ (1.0 - 0.5),\ \ (T)_{0\alpha}^R = 1.5 - \alpha_T\ (1.5 - 1.0),$$
$$(\kappa)_\alpha^L = 1 + \alpha_\kappa \text{ and } (\kappa)_\alpha^R = 3 - \alpha_\kappa . \tag{13}$$

Alpha cuts of fuzzy numbers T_0 and κ are as shown in Table 1.

Table – 1 Alpha cut values of T_0 and κ
[T_O^L, T_O^R] and [κ^L, κ^R] are (L-R) representations of T_O and κ

Alpha	To^L	To^R	κ^L	κ^R
0	0.5	1.5	1	3
0.1	0.55	1.45	1.1	2.9
0.2	0.6	1.4	1.2	2.8
0.3	0.65	1.35	1.3	2.7
0.4	0.7	1.3	1.4	2.6
0.5	0.75	1.25	1.5	2.5
0.6	0.8	1.2	1.6	2.4
0.7	0.85	1.15	1.7	2.3
0.8	0.9	1.1	1.8	2.2
0.9	0.95	1.05	1.9	2.1
1	1	1	2	2

Analytically, fuzzy solution of temperature T(x,t) at x = 0.5 and t = 1 sec from equation (9) with the fuzziness in κ and T_0, can be now written as

$$T\ (x,t) = \tilde{T}_0 \left[1 - erf\left(\frac{0.5}{2} \sqrt{\frac{1}{\tilde{\kappa}}} \right) \right] \tag{14}$$

Equation (14) based on fuzzy vertex method with the alpha level representation of fuzzy parameters (T_O and κ) results the following analytical equations (15-18) for temperature, T (x= 0.5, t = 1).

$$T_1 = T\ (x,t) = T_0^L \left[1 - erf\left(\frac{0.5}{2} \sqrt{\frac{1}{\kappa^L}} \right) \right] \tag{15}$$

$$T_2 = T\ (x,t) = T_0^L \left[1 - erf\left(\frac{0.5}{2} \sqrt{\frac{1}{\kappa^R}} \right) \right], \tag{16}$$

$$T_3 = T(x,t) = T_0^R\left[1 - erf\left(\frac{0.5}{2}\sqrt{\frac{1}{\kappa^L}}\right)\right], \tag{17}$$

$$T_4 = T(x,t) = T_0^R\left[1 - erf\left(\frac{0.5}{2}\sqrt{\frac{1}{\kappa^R}}\right)\right] \tag{18}$$

Finally, left-right (L-R) bounds of temperature, i.e., lower and upper bounds (minimum and maximum values) of temperature for any alpha cut are computed using $T^L(\alpha) = \min(T_1, T_2, T_3, T_4)$ and $T^R(\alpha) = \max(T_1, T_2, T_3, T_4)$.

Computation of temperature of the rectangular bar is repeated for various spatial distances, starting from x = 1 to 50 cm at time, t = 60 seconds. Minimum, most likely and maximum of the temperature profiles for various spatial distances are shown in figure 4.0. It can be explained from figure 4.0 that at distance beyond 30 cm. all the bounds of the temperature are zero. So, uncertainty of the temperature properly exists up to distance x = 25 cm. and the corresponding bounds at that distance are estimated as [0.01, 0.28] ^oC. Temperature bounds at distance 10 cm from figure 4.0 can be reported as [0.18, 0.90] ^oC. Therefore, these values indicate that uncertainty of the temperature decreases as the distance increases. Similar behavior is reported in terms of measures of fuzziness. In order to prove this feature, computation of uncertainty of the temperature profile is expressed in terms of fuzziness measure of uncertainty. Propagation of fuzziness measure of uncertainty for various spatial distances is as shown in figure 5.0. It can be stated from figure 5.0 that fuzziness of the temperature decreases with the increase of the spatial distance. The reason is that temperature measurement at far distance is more precise compared to near end of the source. The computation of uncertainty of temperature profile is further repeated using FCR algorithm. Results of the variation of temperature profile bounds (lower, most likely and upper) are plotted and shown in figure 6.0. It is obvious from figure 6.0 that interval of the temperature bounds at any spatial distance is smaller compared to that using only alpha cut representation of the fuzzy parameters. Therefore, FCR algorithm reduces the interval and also epistemic uncertainty.

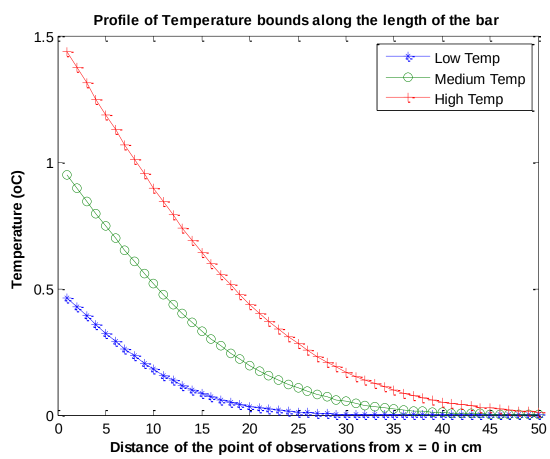

Figure 4. Variation of temperature bounds with most likely value along the length of the system

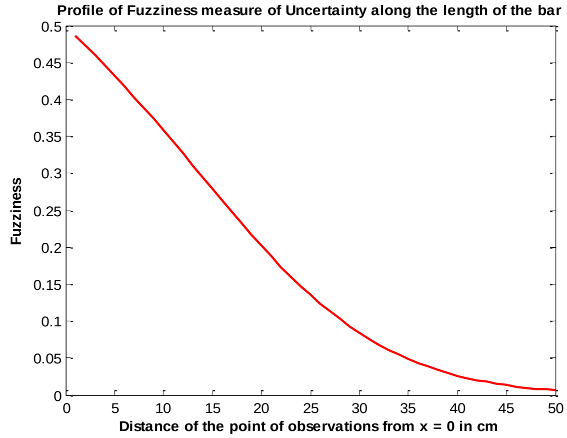

Figure 5. Variation of Measures of Fuzziness of the Temperature

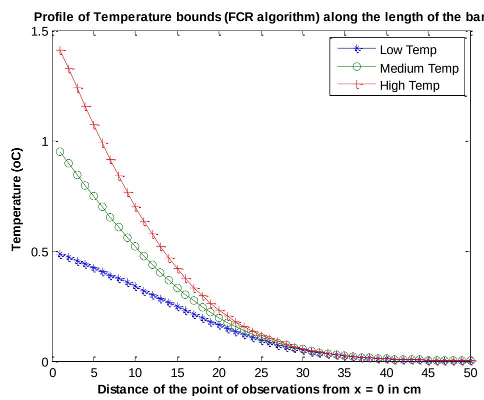

Figure 6. Variation of temperature bounds with most likely value along the length of the system using FCR algorithm

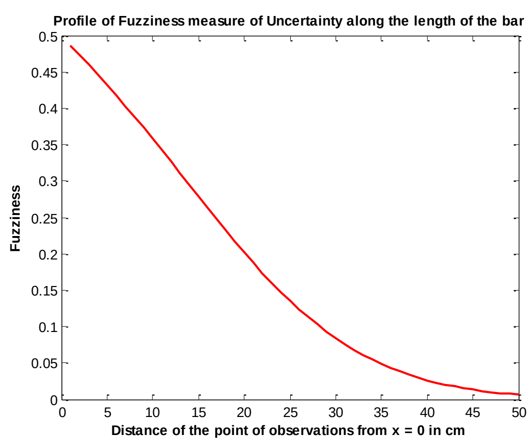

Figure 7. Variation of Measures of Fuzziness of the Temperature with FCR Algorithm

This is the conclusive result of FCR algorithm that reduction of uncertainty is possible by FCR algorithm. Fuzziness measure of uncertainty of temperature using FCR algorithm is shown in figure 7.0 which also indicates that fuzziness decreases with the increase of the spatial distance.

4.2 Case 2: Heat Removable from Fin

In this case, a fin is considered as one dimensional heat transfer problem. One end of the fin is connected to a heat source (whose temperature is known) and heat is allowed to dissipate to the surroundings through the perimeter surface and the free end (tip of the fin). We have computed the uncertainty of temperature and heat removal rate of the fin. The base of the fin is held at a temperature, $T_b = 85^0$C, and the ambient fluid temperature is maintained at $T_a = 25^0$C (figure 8.0(a)). The length of the fin is taken as 2 cm and the diameter is taken as 0.4 cm. The fin is made of copper metal. Membership function of the thermal conductivity (k W/m^{0}C) and the heat transfer coefficient (h W/m^2 ^{0}C) of the material of the fin is as shown in figure 8.0 (b)

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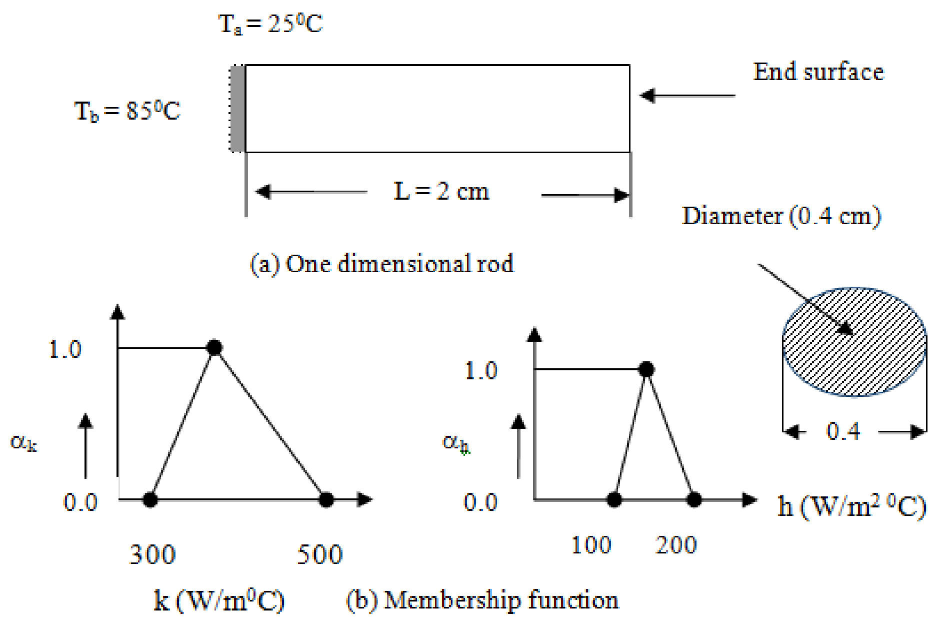

Figure 8.0 Geometry of the Rod and Membership Function of h and k

In this case, the governing differential equation is (from figure 8.0 (a))

$$k.\frac{d^2T}{dx^2} + \dot{q} = 0 \qquad (19)$$

where, k is the thermal conductivity and $\dot{q}$ signifies the strength of the heat source (rate of heat generated per unit volume per unit time). This problem is the steady state problem and hence the necessary boundary conditions are:

$$T(x=0) = T_b \qquad (20)$$

and

$$k\frac{dT}{dx}l_x + h(T - T_a) + \dot{q} = 0 \quad \text{on the surface} \qquad (21)$$

where, l_x is the direction cosine (along x direction) of the outward drawn normal to the boundary. Problem is numerically solved using finite element method. Two noded linear finite elements shown in figure 9.0 are used. Parameters heat transfer coefficient, h and thermal conductivity, k are considered as fuzzy due to their incomplete or sparse measured values. FCR algorithm is used to define their intervals at each alpha level. According to the proposed FCR algorithm, the interval (fuzzy) number is defined as <centre, radius>, where the centre of the fuzzy number $F = [f_1, f_2]$ is defined as $F^0 = \frac{f_1 + f_2}{2}$ = centered value and the radius is defined as $F_R = \frac{f_2 - f_1}{2}$. FCR algorithm details are presented in section 3. Heat removal rate from the fin is given by

$$\begin{aligned} Q_R &= \sum_{e=1}^{2} hpl^{(e)} \left[\frac{T_i + T_j}{2} - T_a \right] \\ &= hpl^{(1)} \left[\frac{T_1 + T_2}{2} - T_a \right] + hpl^{(2)} \left[\frac{T_2 + T_3}{2} - T_a \right] \end{aligned} \tag{22}$$

where, p = perimeter of the surface, $l^{(e)}$ = length of the element, e, T_i, T_j = temperature at the i^{th} and j^{th} node, and T_a = ambient temperature.

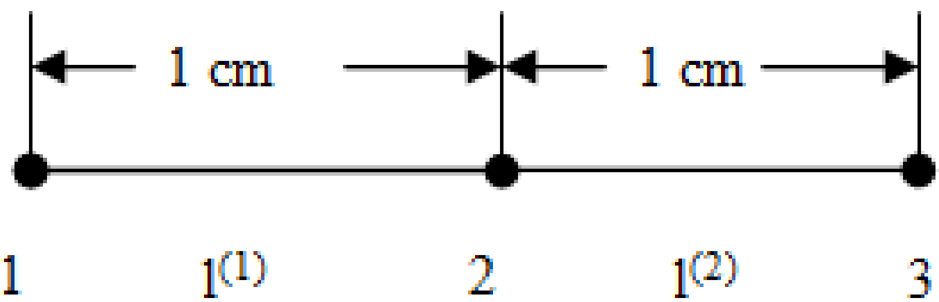

Figure 9. Finite Element Idealizations

The length of each element is taken as $l^{(e)} = 1$ cm . Since D is the diameter of the circular fin, we can write the area of the each section, $A = \frac{3.14 * D * D}{4} = \frac{3.14(0.004)^2}{4} = 1.256 \times 10^{-5} m^2$ and perimeter, $p = 3.14(0.004) = 1.2566 \times 10^{-2} m$. Thermal conductivity, k = [300, 500] W/m ^{0}C and heat transfer coefficient, h = [100, 200] W/m^2 ^{0}C. Existing interval arithmetic rules results a very large interval band, which is reduced using our new FCR algorithm. Usage of fuzziness of the parameters h and k in finite element methodology results finally a matrix equation

$$[\mathrm{K}]\{\mathrm{T}\} = \{F\} \tag{23}$$

where, the matrix , [K] is known as stiffness matrix, {F} represents the load vector and {T} is the solution of the temperature vector. Stiffness matrix [K] is constructed by assembling the matrices $[K^{(1)}]$ and $[K^{(2)}]$, defined for the element 1 and 2 respectively. The matrices $[K^{(1)}]$ and $[K^{(2)}]$ are mathematically formulated as

$$[K^{(1)}] = [K_c^{(1)}] + [K_w^{(1)}], \quad [K^{(2)}] = [K_c^{(2)}] + [K_w^{(2)}]$$

$$[K_c^{(1)}] = \frac{kA}{l^{(1)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad [K_w^{(1)}] = \frac{hPl^{(2)}}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad [K_c^{(1)}] = [K_c^{(2)}],$$

$$[K_w^{(2)}] = \frac{hPl^{(2)}}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + hA\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

The load vector $\{F\}$ in the equation (23) is constructed by assembling $\vec{f}^{(1)}$ and $\vec{f}^{(2)}$ defined for the element (1 and (2) and are formulated as

$$\vec{f}^{(1)} = \frac{hT_a Pl^{(1)}}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \qquad \vec{f}^{(2)} = \frac{hT_a Pl^{(2)}}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + hT_a A\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

It can be easily stated that the elements of the stiffness matrix and the load vector are intervals due to fuzzy parameters heat transfer coefficient, *h* and the thermal conductivity, *k*. Hence traditional interval arithmetic required to solve the system of equations increases the solution interval by a substantial amount. This is due to the inherent arithmetic structure of the interval mathematics. Hence, FCR algorithm is used here to have a consistency in the interval solution of the temperature of the fin. Algorithm is implemented into a FORTRAN program 'FUZZYFIN'. Equation (23) is solved to obtain the temperature output. Membership function of the computed temperatures at nodes 2 and 3 and that of the heat removal rate are shown in figures 10.0 - 12.0. Uncertainty of temperature at node 2 at alpha cut of 0.1 from figure 10.0 can be noted as [81.42, 81.92] ^{o}C and the same at node 3 can be noted from figure 11.0 as [80.09, 80.78]^{o}C. Dividing the interval of temperature bounds at node 2 and at node 3 by the corresponding most likely value (value of temperature for membership value equal to 1) we obtain the coefficient of variation at node 2 as 0.006 and that at node 3 is 0.008. Therefore, it can be concluded that uncertainty of temperature expressed as coefficient of variation at alpha cut level of 0.1 at node 2 is less than that at node 3. Uncertainty (coefficient of variation) of heat removal rate at alpha cut equal to 0.1 from figure 12.0 is similarly observed as 0.60. Measures of fuzziness, i.e., fuzziness measure of uncertainty of temperatures at node 2 and at node 3 are computed using equation (10) and these values are less than zero indicating that at intermediate distance of 1 cm apart from the position of left boundary (temperature here at 85 ^{o}C) fuzziness vanishes. The fuzziness measure of uncertainty of heat removal rate is computed as 0.71 and this value is comparable with the uncertainty of the heat removal rate when expressed in terms of coefficient of variation. Hence, fuzziness measure of uncertainty can be used as measure of non-probabilistic uncertainty.

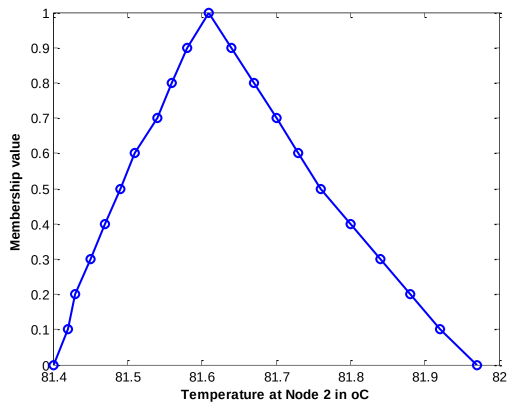

Figure 10. Membership function of temperature at Node 2

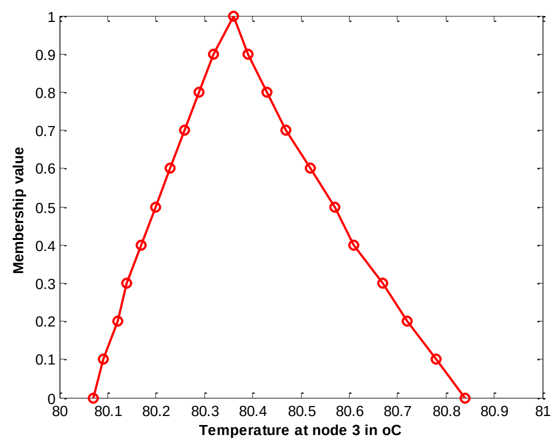

Figure 11. Membership function of temperature at Node 3

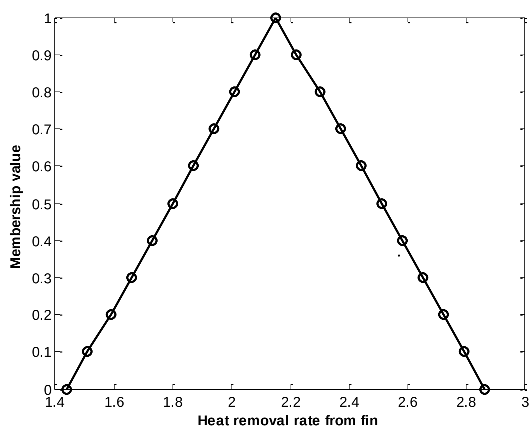

Figure 12. Membership Function of Heat Removal Rate

5. Conclusion

This paper demonstrates the strength and applicability of the fuzzy set theory for accounting the non-probabilistic uncertainty of the analytical and numerical solution of the partial differential equation. Finite element method with fuzzy parameter provides the scope of further research into the Fuzzy Finite Element method (FFEM). FCR algorithm is proposed as one of the technique to reduce the uncertainty. Fuzziness of a fuzzy set can be used as a measure of the non-probabilistic uncertainty. Uncertainty according to classical probability theory is reported as 95% confidence interval estimate of the quantity of interest. At par with this probability theory, coefficient of variation at alpha cut equal to 0.1 can also provide uncertainty of the quantity of interest. The quantification of non-probabilistic uncertainty can be also possible by trapezoidal and other membership function (e.g., trapezoidal, Gaussian, etc.). Future research in this direction will be to quantify the epistemic or non-probabilistic uncertainty using evidence theory.

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