The Theory of Fuzzy Sets:
Beliefs and Realities

Hemanta K. Baruah

Department of Statistics, Gauhati University, Assam, India.
hemanta_bh@yahoo.com

Abstract

On two important counts, the Zadehian theory of fuzzy sets urgently needs to be restructured. First, it can be established that for a normal fuzzy number $N = [\alpha, \beta, \gamma]$ with membership function $\Psi_1(x)$, if $\alpha \leq x \leq \beta$, $\Psi_2(x)$, if $\beta \leq x \leq \gamma$, and 0, otherwise, $\Psi_1(x)$ is in fact the distribution function of a random variable defined in the interval $[\alpha, \beta]$, while $\Psi_2(x)$ is the complementary distribution function of another random variable defined in the interval $[\beta, \gamma]$. In other words, every normal law of fuzziness can be expressed in terms of two laws of randomness defined in the measure theoretic sense. This is how a normal fuzzy number should be constructed, and this is how partial presence of an element in a fuzzy set has to be defined. Hence the measure theoretic matters with reference to fuzziness have to be studied accordingly. Secondly, the field theoretic matters related to fuzzy sets are required to be revised all over again because in the current definition of the complement of a fuzzy set, fuzzy membership function and fuzzy membership value had been taken to be the same, which led to the conclusion that the fuzzy sets do not follow the set theoretic axioms of exclusion and contradiction. For the complement of a normal fuzzy set, fuzzy membership function and fuzzy membership value are two different things, and the complement of a normal fuzzy set has to be defined accordingly. We shall further show how fuzzy randomness should be explained with reference to two laws of randomness defined for every fuzzy observation so as to make fuzzy statistical conclusions. Finally, we shall explain how randomness can be viewed as a special case of fuzziness defined in our perspective with reference to normal fuzzy numbers of the type $[\alpha, \beta, \beta]$. Indeed every probability distribution function is a Dubois-Prade left reference function, and probability can be viewed in that way too.

Keywords: *Randomness-fuzziness consistency principle, field of fuzzy sets, fuzzy randomness, theory of probability.*

1. Introduction

Establishing a new theory is tough; making changes in an existing theory is tougher, particularly when the changes are suggested in a field that had originated nearly half a century ago, and hundreds of books and thousands of articles have meanwhile been published in that field the world over. In this article, we are going to point out two changes that need to be incorporated in the Zadehian theory of fuzzy sets [1]. First, it had been accepted that the fuzzy sets do not in any way conform to the classical measure theoretic formalisms. Secondly, it had been agreed upon that given a fuzzy set neither its intersection with its complement is the null set, nor its union with the complement is the universal set. These two definitions have given rise to a lot of results that defy common sense. In fact, in the Zadehian theory of fuzzy sets, not always logic was followed by mathematics, it had mostly been the other way around.

In course of time, the following two things continued to happen. First, certain fuzzy measures have been defined which are in no way generalizations of any classical measure. Further, certain measures, such as the Hartley like measures for example, have been defined by integrating functions of the membership function of a fuzzy number. Secondly, with a wrong definition of the complement of a fuzzy set, a lot of applications have been done, and studies in fuzzy logic proceeded in a very strange manner. Indeed the theory of fuzzy sets should naturally have been a generalization of the classical theory of sets. Due to the reason of defining the set operation of complementation in the way that has been followed till this day, the theory of fuzzy sets has ended up being something rather unearthly, very different from what mathematics should actually look like. This sort of things has actually divided the world of mathematics into two parts: one part consisting of those who work in this field, and a second part consisting of those who do not really believe a word of it! Perhaps, the present author alone forms an insignificant third part. We have been continuing to apply simple fuzzy arithmetic operations, just plus-minus-product-division and nothing beyond that, in analyzing effect of partial presence in certain data analytical matters that are rather unimportant in our own eyes, without actually believing most of the existing theory of fuzziness. Fuzzy arithmetic using the method of α-cuts is one thing that is perfectly correct. Beyond that, most of the things need to be restructured. In particular, both in theory and in applications, in all matters where the definition of complement of a fuzzy set has been used are in our eyes totally unacceptable.

In this expository article, we are going to reintroduce the Zadehian theory of fuzzy sets rejecting certain assumptions, making it logically sound thereby. We are not interested in modifying anything. We are interested in correcting a few blunders. We know that it would be very difficult to convince those who have already published a lot in this field without ever bothering to think that in their kind of mathematics there is very little logic. In fact, a huge amount of unnecessary publications have been made in this branch of mathematics. Based on wrong axioms, one can definitely build up formalisms that might actually look very much like mathematics. We would like to refute hereby all such illogical results published in the name of the mathematics of fuzziness so far. We understand that correcting a simple error that might have been made recently is one thing, and challenging a multitude of blunders that have been made to continue for nearly half a century is quite another. We do not at all expect that those who have been working in this field would immediately agree with us. In fact, most of them might never agree, because what we are now going to put forward are diametrically opposite of what they have been believing as true so far, and therefore as soon as they decide to agree with us, they would have to reject their own lifetime findings, which is something hardly any human being can possibly be expected to do. After all, if an examiner faces a situation that he would lose his diploma as soon as he allows a particular examinee to get through, it is obvious that the unfortunate examinee might never get through at all! We hope, the new generation of workers would see reason, and would come forward to reconstruct the mathematics of fuzziness anew. Regarding fuzziness, what we are going to express in this article are mathematical realities based on pure logic. Our findings mentioned herein are not based on some popular beliefs. We are not going to say anything differently; we are going to say something different that would ultimately converge to the existing definition of fuzziness. Perhaps such a situation did never arise in the world of mathematics in modern times at least, when an entire theory nearly half a century old is being challenged by one single individual who consistently refused to follow the leader.

We had in the mid-nineties of the last century, independently and without any reference to fuzziness, tried to frame the mathematics of *partial presence* of an element in a set in the

following perspective. When we overwrite, the overwritten portion looks darker for multiple representations. The operation of union of sets does not explain multiple representations of elements. So we defined a set operation called *superimposition of sets*. Then we observed that to explain the presence of different shades of darkness in the umbra and in the penumbra regions of the shadow on a screen when an opaque body is placed in front of a source of light, we need to define partial presence of elements in a set. So from multiple representations, to arrive at partial presence of elements, we quantified the level of maximum darkness in the shadow as unity, so that if the level of presence of darkness in the umbra portion is taken as 1, then that in the penumbra portion would have to be ½. That way too, the Zadehian definition of fuzziness could be arrived at. Thereafter, coming back to the example of overwriting, we observed that if we superimpose *n* equally fuzzy intervals each with membership *1/n*, the standard formalisms of Order Statistics come into play automatically. We then found that what had been already known as a *normal* law of fuzziness could be explained with the help of two laws of randomness that could be defined using a classical theorem on order statistics. We could thus naturally arrive at defining fuzziness with the help of two laws of randomness. We did not have to ponder over how to express randomness in terms of fuzziness; there was no need to do that. In our case, we did not start with an idea that the concept of fuzziness is a competitor of the concept of randomness. The standpoint of the mathematical workers in this field had always been, and still is, that a given law of fuzziness could possibly be linked with *one* law of randomness, and that is how they have been trying without success to link randomness with fuzziness till this day. We found that trying to impose one law of randomness over an interval on which a law of fuzziness had been defined was downright illogical. Ours was clearly a different standpoint. It is obvious that to arrive at our conclusion, one has to look into the matters through the spectacles of the operation of superimposition of sets, which was not there in the literature at that time. Further, one must also know that there exists a classical result called the Glivenko – Cantelli theorem on Order Statistics. The Glivenko – Cantelli theorem deals with establishing a limiting probability law from an empirical probability law. This theorem, like all other probabilistic formalisms, is true in the broader measure theoretic sense too of defining randomness. Unfortunately, at that juncture, the question of examiners losing their diplomas if a particular examinee is allowed to get through came into picture, and we decided without much fuss to publish our findings locally so as to keep a claim should the need arise later [2]. In every strife between scientific logic forwarded by one single individual and baseless belief coupled with collective arrogance of all others concerned, logic had always been the instantaneous loser. This happened many a time in the history of science. In fact, here the Gödelian theorem of incompleteness arguing that no logical system itself can ever prove that it is true came up [3]. While remaining within a system, it is impossible to disprove an axiom included in that system. So as to find a flaw in a logical system, one has to look into the system from a perspective outside it, and to appreciate a logical flaw in a system, one therefore has to come out of the system first. We started with the idea of superimposition of sets with constant partial presence, and arrived at the Zadehian definition of fuzzy sets. Nowhere did we use any Zadehian axiom while defining our randomness – fuzziness consistency principle. We deduced our results remaining wholly outside the existing system. So while in our eyes the assumption of being able to infer a law of randomness from a law of fuzziness was absolutely absurd, in the eyes of those who had been working on fuzziness it was obviously difficult to digest that a normal law of fuzziness is rooted at two laws of randomness. Indeed the Gödelian definition of incompleteness indirectly says that one can not judge anything beyond one's knowledge, and knowledge is always incomplete because newer axioms invariably continue to arrive. Nearly

a decade later, when we used our operation of set superimposition towards recognizing calendar based periodic patterns, we were very highly bemused to observe that our findings got positive nod from the examiners ([see for example [4]). This time, the examiners were from a different system. We there used the definition of *superimposition* of constantly fuzzy sets, and they readily agreed that there was nothing wrong in doing so. The readers can verify for themselves that this set operation proposed by us way back in 1999 has been in use since 2008 in pattern recognition without any objection from anyone. That settled the issue that our definition of set superimposition was indeed firmly footed. To define partial presence, we thereafter used the Glivenko – Cantelli theorem on superimposed constantly fuzzy intervals, and this theorem is something very classical in the statistical literature. In other words, we understood that our definition of partial presence of an element in a set, better known as fuzziness since 1965, must necessarily be correct. Here then is an example of a theory that was declared unacceptable by those who had been preaching something that was precisely the opposite of what the theory wanted to describe, but an application of a part of that same theory was found to be acceptable by another set of people who were not insiders of that mathematical cult, more than a decade later. One conclusion was evident. If we were found correct in 2008, we must have been correct more than a decade earlier too! This is why we now have decided to reappear in the scene with a view to suggesting a complete restructuring of the existing mathematics of fuzziness. The readers may perhaps note that for an original piece of mathematical research, unlike the researches in any other branch of science and technology, one hardly needs anything more than a creative mind, and one's creativity does not depend on the pedestal from which one speaks, it never did.

At this point, we would like to remind the readers one important but generally misunderstood point regarding the definition of randomness. The notion of probability does *not* enter into the definition of a random variable (See for example [5], page 43). Therefore in this article whenever we would refer to a law of randomness, we would mean so in the measure theoretic sense. What we mean is, when a variable is probabilistic, it has to be random by definition, although when a variable is random, it need not be probabilistic in the statistical sense. Accordingly, all results of the classical theory of probability are automatically applicable to a random variable defined in the measure theoretic sense.

Thereafter, we found that the operation of complementation of a normal fuzzy set as defined in the Zadehian theory of fuzziness does not explain the principles of exclusion and contradiction followed by the classical sets. We observed that in the Zadehian definition of complementation, membership value and membership function had been taken to be of the same meaning. When a six feet tall man stands on a three feet tall table, his height does not suddenly become nine feet, because in the later case his height has to be measured not from the ground but from the table. Similarly, if a glass is half full, it is also half empty but the empty portion has to be counted from ½, and not from zero. This simple reasoning helped us to see that defining membership with respect to a value of reference could solve the problem, that of following the principles of exclusion and contradiction by fuzzy sets. Once again, the Gödelian principle of one not being able to judge anything beyond one's knowledge came into picture, and we decided once again, this time without any fuss, to publish our findings locally [6].

The discovery of fuzzy sets was an epoch making event in the history of mathematics. It actually led to a paradigm shift in data analytical matters. But the fuzzy mathematics fraternity has to realize that one can arrive at the definition of anything, fuzziness for example, starting from a different perspective too. In our case, we observed that partial presence of an element in a set can actually be *seen*, for instance in the case of the umbra-

penumbra example mentioned earlier. After that we proceeded to explain partial presence mathematically. It was another matter that unfortunately for us what we arrived at in the mid-nineties was a concept already known as fuzziness since the mid-sixties. Our formalisms are indeed totally different. However, if that happens to explain the measure theoretic matters and certain other associated formalisms related to fuzziness logically, then most of the earlier results would have to be thrown out first, because they are mostly based on some rootless mathematics that has been in turn made to stand on certain illogical axioms. The sooner we realize this, the better. In fact, modifications and generalizations of earlier results regarding fuzziness in the name of research have blinded the mathematical workers to such an extent that in most of the cases they would actually be unable to explain the physical significance of their own results. In something that looks very much like mathematics, devoid of any logic whatsoever, it would certainly be impossible to explain the physical significance of the concerned symbolic expressions. Physical significance must be given proper credence in mathematical researches. Otherwise, tomorrow someone might come up with an insane idea of defining a triangular quadrilateral for example, and perhaps there would be no dearth of people to generalize even that kind of an absolute nonsense without at all trying to understand any head or tail of what they are doing. We earnestly hope, the mathematics fraternity in general and the statistics fraternity in particular would eventually come forward and look into what really has been going on in the name of development in the world of fuzzy mathematics.

Before proceeding further, we would like the readers to note a basic mathematical matter. Given any differentiable function of a continuous variable, one can always proceed to differentiate it with respect to the variable concerned. Similarly, one can also proceed to integrate that function with respect to the variable. Things of that sort are actually done as classroom exercises everywhere. But in reality, when one either differentiates or integrates a function, there must necessarily be some physical significance of the results thus found. For example, integrating a probability distribution function is totally meaningless, because a probability distribution function defines an area, and integrating, and therefore finding the area under a function already defining an area is a meaningless exercise. In the case of trying to infer a law of randomness from an existing law of fuzziness in the name of framing a probability – possibility consistency principle, this kind of a logically meaningless exercise was done, and such things have been unfortunately going on unabated in the name of the mathematics of fuzziness till this day.

In what follows, we shall put forward a mathematical explanation of how partial presence of an element in a fuzzy set originates. This will lead us to the missing link between randomness and fuzziness. We would then give two counterexamples to show that the Zadehian definition of complement of a fuzzy set is *not* correct. Thereafter we shall put forward the actual definition of the complement of a fuzzy set. We would then give an explanation of what fuzzy randomness should actually mean. Finally, we shall explain the only possible way how randomness can be viewed as a special case of fuzziness.

2. The mathematical explanation of partial presence

A fuzzy real number [α, β, γ] is an interval around the real number β with the elements in the interval being partially present. However this definition does not automatically say anything about the mathematical explanation of partial presence of an element in a set. Partial presence of an element in a fuzzy set has been defined by the name *membership function*. A *normal* fuzzy number $N = [\alpha, \beta, \gamma]$ is thus associated with a membership function $\mu_N(x)$, where $\mu_N(x)$ is $\Psi_1(x)$, if $\alpha \leq x \leq \beta$, is $\Psi_2(x)$, if $\beta \leq x \leq \gamma$, and is 0, otherwise. Here $\Psi_1(x)$ is

continuous and nondecreasing in the interval $[\alpha, \beta]$, and $\Psi_2(x)$ is continuous and nonincreasing in the interval $[\beta, \gamma]$, with $\Psi_1 (\alpha) = \Psi_2 (\gamma) = 0$, $\Psi_1 (\beta) = \Psi_2 (\beta) = 1$. In the now classical Dubois – Prade nomenclature, $\Psi_1(x)$ is called the Left Reference Function, and $\Psi_2(x)$ is called the Right Reference Function of the normal fuzzy number. Even after correct identification of the properties of the reference functions, the question as to wherefrom such functions originate remained unanswered. In other words, how exactly to construct a fuzzy number still remained a question. In the books concerned, various types of fuzzy membership functions following the Dubois – Prade definition are discussed. However, nothing has yet been said as to why some particular membership function should be used in any situation in preference to other such functions. For example, in the applications the triangular fuzzy number is said to be used for its simplicity. That kind of an explanation is logically incomplete. Wherefrom does this so called simplicity of the triangular fuzzy numbers originate? We need to answer that too.

We would like to start the proceedings with the following question: is it possible to get a law of randomness from a given law of fuzziness? The answer is, no, it is not possible. All attempts by workers to do so have simply failed. We now ask another question: given two laws of randomness, one on $[\alpha, \beta]$ and the other on $[\beta, \gamma]$, would it be possible to define a law of fuzziness on $[\alpha, \beta, \gamma]$? The answer we found was: yes, this is actually possible. Defining the operation called Superimposition of Sets [2] and using the Glivenko – Cantelli Theorem ([7], page 20) on Order Statistics, the present author ([8], [9], and [10]) has established the following result which we shall now state as a theorem that uncovers the missing link between fuzziness and randomness, which was being searched for by the workers in fuzziness since 1965.

Theorem -1: For a normal fuzzy number $N = [\alpha, \beta, \gamma]$ with membership function $\mu_N(x) = \Psi_1(x)$, if $\alpha \leq x \leq \beta$, $= \Psi_2(x)$, if $\beta \leq x \leq \gamma$, and $= 0$, otherwise, such that $\Psi_1 (\alpha) = \Psi_2 (\gamma) = 0$, $\Psi_1 (\beta) = \Psi_2 (\beta) = 1$, $\Psi_1(x)$ is the distribution function of a random variable defined in the interval $[\alpha, \beta]$, and $\Psi_2(x)$ is the complementary distribution function of another random variable defined in the interval $[\beta, \gamma]$.

For easy readability of this article, we now proceed to describe our standpoint very concisely. We defined ([2], [8], [9], [10]) the operation of superimposition of two real intervals $[a_1, b_1]$ and $[a_2, b_2]$ as $[a_1, b_1]$ (S) $[a_2, b_2] = [a_{(1)}, a_{(2)}] \cup [a_{(2)}, b_{(1)}]^{(2)} \cup [b_{(1)}, b_{(2)}]$ where $a_{(1)} = \min (a_1, a_2)$, $a_{(2)} = \max (a_1, a_2)$, $b_{(1)} = \min (b_1, b_2)$, and $b_{(2)} = \max (b_1, b_2)$. Here we have assumed without loss of any generality that $[a_1, b_1] \cap [a_2, b_2]$ is not void, or in other words that $\max (a_i) \leq \min (b_i)$, i = 1, 2.
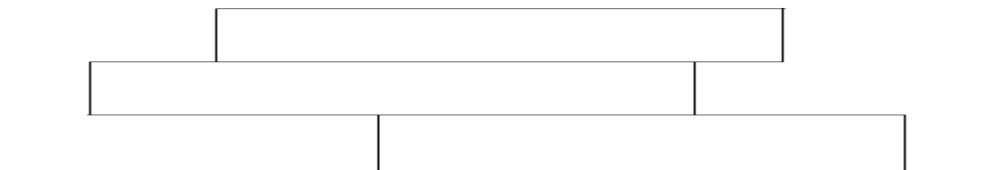

Figure. 1. Superimposition of $[x_1, y_1]^{(1/3)}$, $[x_2, y_2]^{(1/3)}$ and $[x_3, y_3]^{(1/3)}$

Figure. 2. Cumulative and complementary cumulative distribution functions

Figure. 3. Discrete Dubois-Prade left and right reference functions

We would like to explain the matters with the help of diagrams. For the three *intervals* $[x_1, y_1]^{(1/3)}$, $[x_2, y_2]^{(1/3)}$ and $[x_3, y_3]^{(1/3)}$ all with elements with a constant level of partial presence equal to 1/3 everywhere, we shall have $[x_1, y_1]^{(1/3)}$ (S) $[x_2, y_2]^{(1/3)}$ (S) $[x_3, y_3]^{(1/3)} = [x_{(1)}, x_{(2)}]^{(1/3)}$ U $[x_{(2)}, x_{(3)}]^{(2/3)}$ U $[x_{(3)}, y_{(1)}]^{(1)}$ U $[y_{(1)}, y_{(2)}]^{(2/3)} [y_{(2)}, y_3]^{(1/3)}$, where, for example, $[y_{(1)}, y_{(2)}]^{(2/3)}$ represents the interval $[y_{(1)}, y_{(2)}]$ with level of partial presence 2/3 for all elements in the entire interval, $x_{(1)}, x_{(2)}, x_{(3)}$ being values of x_1, x_2, x_3 arranged in increasing order of magnitude, and similarly $y_{(1)}, y_{(2)}, y_{(3)}$ being values of y_1, y_2, y_3 arranged in increasing order of magnitude again. We here presumed that $[x_1, y_1] \cap [x_2, y_2] \cap [x_3, y_3]$ is not void.

Consider now the Figs.1, 2 and 3 shown above. In Fig. 1 superimposition of the intervals $[x_1, y_1]^{(1/3)}$, $[x_2, y_2]^{(1/3)}$ and $[x_3, y_3]^{(1/3)}$ all with elements with a constant level of partial presence equal to 1/3 everywhere has been depicted. That leads us to Fig. 2 in which we have depicted the observed cumulative distribution function defined on $[x_{(1)}, x_{(2)}]$, $[x_{(2)}, x_{(3)}]$ and $[x_{(3)}, y_{(1)}]$, and the observed complementary cumulative distribution function on $[x_{(3)}, y_{(1)}]$, $[y_{(1)}, y_{(2)}]$ and $[y_{(2)}, y_3]$, that can be seen to have originated due to superimposition of the intervals $[x_1, y_1]^{(1/3)}$, $[x_2, y_2]^{(1/3)}$ and $[x_3, y_3]^{(1/3)}$. In Fig. 3, we are depicting the Dubois–Prade left and right reference functions arising due to superimposition of the three equally fuzzy intervals each with level of partial presence of elements equal to 1/3 in $[x_1, y_1]^{(1/3)}$, $[x_2, y_2]^{(1/3)}$ and $[x_3, y_3]^{(1/3)}$.

One can clearly see from Figs. 2 and 3 that the left reference function is actually the cumulative distribution function of a random variable, and that the right reference function is the complementary cumulative distribution function of another random variable.

From the diagrams, one point gets very clearly reflected. If we increase the number of intervals, with partial presence of every element for every interval being the inverse of the number of intervals, two laws of randomness should lead to one law of fuzziness. We can see that formalisms of order statistics would now come into play automatically, and to deal with empirical probability distribution functions we already have the Glivenko-Cantelli theorem, application of which should now lead to the conclusion that superimposition of an infinite number of intervals, with level of partial presence of the elements in every interval tending to zero, would define a fuzzy number.

Consider now two probability spaces (Ω_1, A_1, Π_1) and (Ω_2, A_2, Π_2) where Ω_1 and Ω_2 are real intervals $[\alpha, \beta]$ and $[\beta, \gamma]$ respectively. Let $x_1, x_2, \ldots, x_n$, and $y_1, y_2, \ldots, y_n$, be realizations

in $[\alpha, \beta]$ and $[\beta, \gamma]$ respectively. So for n intervals $[x_1, y_1]^{(1/n)}$, $[x_2, y_2]^{(1/n)}$, ..., $[x_n, y_n]^{(1/n)}$ all with elements with a constant level of partial presence equal to $1/n$ everywhere, we shall have

$$[x_1, y_1]^{(1/n)} (S) [x_2, y_2]^{(1/n)} (S) \ldots\ldots (S) [x_n, y_n]^{(1/n)}$$
$$= [x_{(1)}, x_{(2)}]^{(1/n)} U [x_{(2)}, x_{(3)}]^{(2/n)} U \ldots\ldots U [x_{(n-1)}, x_{(n)}]^{((n-1)/n)} U [x_{(n)}, y_{(1)}]^{(1)} \quad (1)$$
$$U [y_{(1)}, y_{(2)}]^{((n-1)/n)} U \ldots\ldots U [y_{(n-2)}, y_{(n-1)}]^{(2/n)} U [y_{(n-1)}, y_{(n)}]^{(1/n)},$$

where, for example, $[y_{(1)}, y_{(2)}]^{((n-1)/n)}$ represents the interval $[y_{(1)}, y_{(2)}]$ with level of partial presence $((n-1)/n)$ for all elements in the entire interval, $x_{(1)}, x_{(2)}, \ldots\ldots, x_{(n)}$ being values of $x_1, x_2, \ldots\ldots, x_n$ arranged in increasing order of magnitude, and $y_{(1)}, y_{(2)}, \ldots\ldots, y_{(n)}$ being values of $y_1, y_2, \ldots\ldots, y_n$ arranged in increasing order of magnitude again. Define now

$$\Phi_n(x) = 0, \text{ if } x < x_{(1)},$$
$$= (r-1)/n, \text{ if } x_{(r-1)} \leq x \leq x_{(r)}, r = 2, 3, \ldots, n,$$
$$= 1, \text{ if } x \geq x_{(n)};$$

$\Phi_n(x)$ here can be seen to be an empirical distribution function for which the underlying theoretical distribution function is $\Phi(x)$. Now the Glivenko-Cantelli theorem states that $\Phi_n(x)$ converges to $\Phi(x)$ uniformly in x. This means,

$$\text{Sup} \left| \Phi_n(x) - \Phi(x) \right| \to 0.$$

Application of this theorem on the intervals $[\alpha, \beta]$ and $[\beta, \gamma]$ separately proves Theorem-1 stated above.

Our standpoint of defining a normal fuzzy number does not defy the Dubois – Prade nomenclature. It is known that a distribution function of a random variable is non-decreasing, and that a complementary distribution function of a random variable is non-increasing. The functions are continuous and differentiable. Observe that integration of a distribution function does not make any logical sense, hence trying to infer anything out of integration of such a function is meaningless. In other words, finding the area under the curve $\mu_N(x)$ is of no logical meaning. On the other hand, differentiation of $\Psi_1(x)$ and $(1 - \Psi_2(x))$ would give us two density functions. This means, we need two laws of randomness, one in the interval $[\alpha, \beta]$ and the other in $[\beta, \gamma]$, to construct a normal fuzzy number $[\alpha, \beta, \gamma]$. For a triangular fuzzy number, differentiation of $\Psi_1(x)$ and $(1 - \Psi_2(x))$ would give us two uniform density functions. It is well known that the uniform law of randomness is the simplest of all probability laws. Thus two uniform laws of randomness lead to the simplest fuzzy number. Computational simplicity notwithstanding, simplicity of the triangular fuzzy number is thus rooted at the simplicity of two uniform laws of randomness.

While dealing with uncertainty, we have to observe one pertinent point. We first have to see whether it is plain randomness, in which case the statistical tools would be sufficient to deal with the situation. For example, if we know that some random error following an appropriate probability law can explain a situation concerned, there already is statistical mathematics to govern that. On the other hand, if we have a situation in which there are two laws of randomness in action at least theoretically, then we indeed have a fuzzy situation. For example, minimum and maximum temperatures in any place could be two random variables. Temperature in any place can therefore be taken as a fuzzy variable. From observations on minimum and maximum temperatures for an appropriate number of days, we can infer about the theoretical probability laws defined with reference to minimum and maximum temperatures, and we can accordingly construct a fuzzy number. Similarly, the price of perishable goods may have maximum and minimum values everyday. In this situation too, we would end up in getting fuzziness that can be defined with the help of two laws of randomness.

Theorem -1 leads us to the following principle:

The Randomness - Fuzziness Consistency Principle: For a normal fuzzy number of the type $N = [\alpha, \beta, \gamma]$ with membership function $\mu_N(x) = \Psi_1(x)$, if $\alpha \leq x \leq \beta$, $= \Psi_2(x)$, if $\beta \leq x \leq \gamma$, and $= 0$, otherwise, with $\Psi_1(\alpha) = \Psi_2(\gamma) = 0$, $\Psi_1(\beta) = \Psi_2(\beta) = 1$, the partial presence of a value x of the variable X in the interval $[\alpha, \gamma]$ is expressible as

$$\mu_N(x) = \theta \, \text{Prob} \, [\alpha \leq X \leq x] + (1 - \theta) \, \{1 - \text{Prob} \, [\beta \leq X \leq x]\},$$

where Prob $[\alpha \leq X \leq x] = \Psi_1(x)$, if $\alpha \leq x \leq \beta$, Prob $[\beta \leq X \leq x] = 1 - \Psi_2(x)$, if $\beta \leq x \leq \gamma$, with $\theta = 1$, if $\alpha \leq x \leq \beta$, and $= 0$, if $\beta \leq x \leq \gamma$.

In other words, the membership function explaining a fuzzy variable taking a particular value is either the distribution function of a random event or the complementary distribution function of another random event. Hence, partial presence of an element in a fuzzy set can actually be expressed either as a distribution function or as a complementary distribution function. As an application of this principle, consider a fuzzy number $X = [a, b, c]$. Let a function of X, $f(X) = [f(a), f(b), f(c)]$ be another fuzzy number. Let the density functions with respect to the distribution functions $\Psi_1(x)$ and $(1 - \Psi_2(x))$ be $\varphi_1(x)$ and $\varphi_2(x)$. If $y = f(x)$ can be written as $x = g(y)$, let $dx/dy = \xi(y)$. Now replacing x by $g(y)$ in $\varphi_1(x)$ and $\varphi_2(x)$ we obtain $\varphi_1(x) = \psi_1(y)$ and $\varphi_2(x) = \psi_2(y)$, say. Then the membership function of $f(X)$ would be given by

$$\mu_{f(X)}(x) = \int_{f(a)}^{x} \{\psi_1(y)\,\xi(y)\} \, dy, f(a) \leq y \leq f(b),$$
$$= \int_{f(b)}^{x} \{\psi_2(y)\,\xi(y)\} \, dy, f(b) \leq y \leq f(c),$$
$$= 0, \text{otherwise.}$$

We have verified that this method returns the same membership functions which can be found by using the standard method of α – cuts available in the literature on fuzziness [11]. We have further verified that all sorts of fuzzy arithmetic can easily be done using our method of looking at the membership function either as a distribution function or as a complementary distribution function whichever is the case [12]. Now that we know that a law of normal fuzziness can be defined with reference to two laws of randomness, we are in a position to use probabilistic mathematics to draw inferences regarding normal fuzziness. For example, Shannon's entropy, better known as Shannon's Diversity Index in the life sciences, can be applied to analyze a fuzzy situation. As we all know, the Shannon Index is maximum when the underlying probability law is uniform, and is minimum when there is no diversity at all in the sense that there is only one outcome with probability 1. This index in the discrete form is however interval sensitive. From the membership function of a normal fuzzy number we would be able to find two Shannon indices, one from the left reference function and one from the right reference function. Such a pair of Shannon indices can then be used for inferring about fuzziness in terms of entropy.

Our randomness – fuzziness consistency principle has one more connotation. For a classical set, an element is always fully present, in the sense that membership of an element is either zero if the element is not in the set, and is unity if the element is in the set. In our sense, if a real number is to be considered as a special case of a fuzzy real number defined around that real number, it is obvious that the probability of occurrence of that number would be equal to 1. In other words, our definition of a fuzzy real number can be seen to be a generalization of the definition of a real number in the probabilistic sense too.

We indeed live in a world of fuzziness, and we have shown that every law of normal fuzziness in turn can be explained with the help of two laws of randomness. Though not in all cases, the statistical definition of randomness would suffice in some cases while defining fuzziness, in the sense that in the other cases the broad measure theoretic definition would

come into play. Randomness thus comes into picture wherever fuzziness exists. Einstein might not have liked to agree, but Hawking rightly assessed the importance of statistical randomness in cosmological matters ([13], page 26). We would like to add that not only in cosmological matters, but also in everything that is fuzzy in Nature, the ultimate underlying decider is nothing but randomness, statistical or otherwise. God does play dice after all!

If our consistency principle is accepted to be true, then the earlier consistency principles must be rejected. Not both of two opposing concepts can simultaneously be true. We have started with two laws of randomness, and logically arrived at the Zadehian concept of fuzziness. The logical truth in this case is that membership of a particular value of a fuzzy variable is expressible either as the value of a distribution function or as the value of a complementary distribution function. Accordingly, trying to find a measure, and therefore defining an index as a function of that, based on the entire membership function, is not logical. In fact, in the earlier consistency principles a law of randomness was assumed over the same space on which a law of fuzziness was defined. We actually need *two* laws, and not just one law, of randomness to define a law of fuzziness. Unfortunately, the mathematics of fuzziness continued to proceed as discussed below.

By using his definition of fuzzy sets, Zadeh [14] noted that the notions of an event and its probability can be extended in a natural fashion to fuzzy events, and he expected that such an extension might eventually enlarge the domain of applicability of probability theory, especially in those fields in which fuzziness is a pervasive phenomenon. Matters related to Borel measurability of the membership function were used to define mean of a fuzzy event and other associated matters such as variance and entropy. Later on too, in his discussions on fuzzy sets with reference to the theory of possibility, Zadeh [15] has clearly mentioned that a fuzzy variable is associated with a possibility distribution in much the same way that a random variable is associated with a probability distribution.

In other words, the unfortunate confusions had started right in the beginning. As soon as one starts with a belief that the notion of randomness could be extended to that of fuzziness, one presumes that over the same interval on which a law of fuzziness can be defined, a law of randomness too can be defined. This means, if there is a probability density function defined on an interval, then that density would be in some way related to a membership function defined on that interval. However, as we have mentioned earlier, while a probability density function can be integrated to find the probability distribution function concerned, integrating the membership function does not make any sense. According to our findings, the left reference function is already a distribution function, and integrating a distribution function is a meaningless exercise. The same kind of an explanation would be applicable in the case of the right reference function too. In fact, it is also to be noted that the use of the word *distribution* in defining possibility distribution was not really in the sense in which it is used in classical measure theory.

Basic measure theoretic matters ensure that integration of a probability density function does have a measure theoretic meaning, while integration of a probability distribution function can not have any measure theoretic meaning. One simple reason cited for that would be enough. A measure has to follow one pertinent point: the measure of a point is *zero*. Possibility of occurrence of a point is defined by the membership function, and therefore possibility of occurrence of a point is not necessarily zero. Hence, enforcing measure theoretic formalisms with reference to the membership function is absolutely illogical.

That was how the confusions had started, and went on unabated. Klir's [16] views in this regard were to extract a law of randomness by scaling the law of fuzziness defined already on an interval. However, the notion of probability should not be forced upon in that way just

because here in this case we have some positive fractions summing up to unity. A probability law has to exist logically, and then we may proceed to infer about it mathematically.

Dubois and Prade (see for example [17]) rightly found Klir's assumptions on transformation of possibility to probability by scaling membership values suitably as debatable. However, they defined a concept of upper probability and named it *possibility measure* [18]. Very recently, Dubois and Hullermeier [19] have even gone for comparing probability measures extracted from the knowledge of possibility. Here too then as we can see, there is this notion of finding *one* law of randomness from *one* normal law of fuzziness defined by left and right reference functions.

As we have seen, the wrong notion of extracting one law of randomness from the knowledge of one law of fuzziness is at the root of the concerned misunderstandings. As long as one would continue to believe that a law of randomness can be established from a law of normal fuzziness, one would continue to keep the mathematics of fuzziness in the middle of nowhere. On one hand, you ridicule randomness as an inappropriate tool to define uncertainty in certain situations, while on the other hand you try to bring in randomness into picture in the name of defining upper probability and such other things; at least mathematics should be spared from such double talk. Yes, not all uncertainties can be explained using randomness, and fuzziness is indeed more prevalent than randomness. But trying to force one law of randomness on a space on which one law of fuzziness has already been found to be at work is downright ridiculous. Mathematics must be based on pure logic, and not on some baseless beliefs.

3. The complement of a fuzzy set

According to the Zadehian definition, if a normal fuzzy number $N = [\alpha, \beta, \gamma]$ is associated with a membership function $\mu_N(x)$, where $\mu_N(x) = \Psi_1(x)$, if $\alpha \leq x \leq \beta$, $= \Psi_2(x)$, if $\beta \leq x \leq \gamma$, and $= 0$, otherwise, the complement N^C will have the membership function $\mu_N{}^C(x)$, where

$$\mu_N{}^C(x) = 1 - \Psi_1(x), \text{ if } \alpha \leq x \leq \beta,$$
$$= 1 - \Psi_2(x), \text{ if } \beta \leq x \leq \gamma, \text{ and}$$
$$= 1, \text{ otherwise.}$$

We first cite two counterexamples that this definition is defective.

Counterexample - 1: First, we would like to ask the readers a simple question. Can a statement and its complement ever be the same? Common sense says that the answer is negative. Consider now the set of real numbers with constant fuzzy membership function equal to ½ everywhere. Therefore according to the Zadehian definition, its complement too will have the constant fuzzy membership function equal to $(1 - \frac{1}{2}) = \frac{1}{2}$ everywhere.

In other words, here is an example of a statement defining a fuzzy number, which is exactly the same as the statement defining its complement! Accordingly, if the Zadehian definition of the complement of a fuzzy set is true, we have arrived at a contradiction that a statement and its complement can be the same. Some people might still argue that a half truth is half wrong too, and therefore they are equivalent! We therefore proceed to cite a second counterexample.

Counterexample - 2: We would like to ask the readers another simple question. Can a statement ever include its complement? Once again, common sense says that the answer is negative. Consider now the set of real numbers with constant fuzzy membership function

equal to ¾ everywhere. Therefore according to the Zadehian definition, its complement will have constant fuzzy membership function equal to $(1 - ¾) = ¼$ everywhere.

In other words, here is an example of a statement that actually includes its complement! Accordingly, if the Zadehian definition of the complement of a fuzzy set is true, we have arrived at a contradiction that a statement can include its complement.

Our counterexamples are based on the fact that if a glass is partially filled with water, then the height of empty portion is to be counted from the height upto which the glass is partially full. These *two* counterexamples should be enough to establish that the very definition of complement of a fuzzy set *is* wrong. In fact, here logic has been forced to follow mathematics. Where is the error then? Fuzzy membership function and fuzzy membership value are two different things. In the Zadehian definition of complementation, these two things have been taken to be the same, and that is where the error lies.
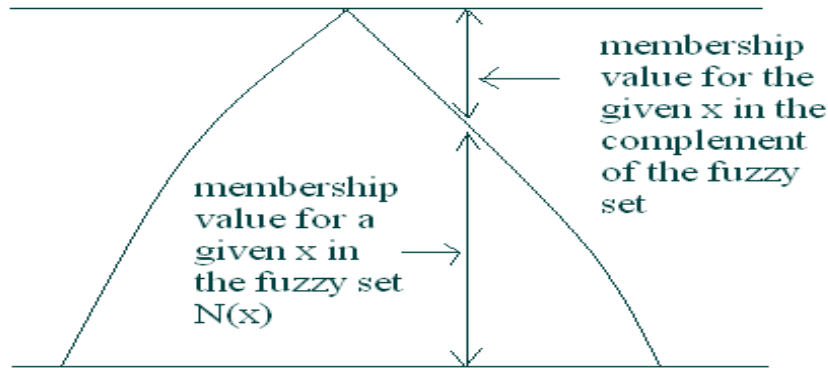

Figure. 4. Complement of a normal fuzzy number

Let $\mu_1(x)$ and $\mu_2(x)$ be two functions, $0 \le \mu_2(x) \le \mu_1(x) \le 1$. For a fuzzy number denoted by $\{x,\ \mu_1(x),\ \mu_2(x);\ x\ \varepsilon\ \Omega\}$, we would call $\mu_1(x)$ the fuzzy membership function, and $\mu_2(x)$ a reference function, such that $\{\mu_1(x) - \mu_2(x)\}$ is the fuzzy membership value for any *x*. The present author [20] has shown that such a definition of a fuzzy set leads to opening the bottleneck that did not allow the fuzzy sets to form a field. The explanation is as follows. In the definition of complement of a fuzzy set, the fuzzy membership *value* and the fuzzy membership *function* have to be different, in the sense that for a usual fuzzy set the membership value and the membership function are of course equivalent. We do not really need our definition to describe the usual fuzzy sets. However, the only way to define complement of a fuzzy set is to use our definition; we do not have any alternative.

Say,

$$A\ (\mu_1, \mu_2) = \{x,\ \mu_1(x),\ \mu_2(x);\ x\ \varepsilon\ \Omega\}$$

and

$$B\ (\mu_3, \mu_4) = \{x,\ \mu_3(x),\ \mu_4(x);\ x\ \varepsilon\ \Omega\}.$$

Then the operations of intersection and union defined naturally as

$$A\ (\mu_1, \mu_2) \cap B\ (\mu_3, \mu_4) = \{x,\ \min\ (\mu_1(x),\ \mu_3(x)),\ \max\ (\mu_2(x),\ \mu_4(x));\ x\ \varepsilon\ \Omega\}$$

and

$$A\ (\mu_1, \mu_2)\ \text{U}\ B\ (\mu_3, \mu_4) = \{x, \max\ (\mu_1(x), \mu_3(x)), \min\ (\mu_2(x), \mu_4(x)); x\ \varepsilon\ \Omega\}$$

leads to the conclusion that for usual fuzzy sets too with $\mu_2(x) = \mu_4(x) = 0$, we get the same results.

For two fuzzy sets $A\ (\mu, 0) = \{x, \mu(x), 0; x\ \varepsilon\ \Omega\}$ and $B\ (1, \mu) = \{x, 1, \mu(x); x\ \varepsilon\ \Omega\}$ defined over the same universe Ω, we would have

$$A\ (\mu, 0) \cap B\ (1, \mu) = \{x, \min\ (\mu(x), 1), \max\ (0, \mu(x)); x\ \varepsilon\ \Omega\}$$
$$= \{x, \mu(x), \mu(x); x\ \varepsilon\ \Omega\}$$

which is nothing but the null set φ. In other words, $B\ (1, \mu)$ is nothing but $(A\ (\mu, 0))^C$ in the sense of classical set theory. This means, if we define a fuzzy set (see Fig. 4)

$$(A\ (\mu, 0))^C = \{x, 1, \mu(x); x\ \varepsilon\ \Omega\},$$

it can be seen that it is nothing but the complement of the fuzzy set

$$A\ (\mu, 0) = \{x, \mu(x), 0; x\ \varepsilon\ \Omega\}.$$

Our definition leads to the assertions that $A \cap A^C = \varphi$, the null set. Further, it can also be shown that $A\ \text{U}\ A^C = \Omega$, the universal set. It can be verified that in our definition the following properties are valid for fuzzy sets A, B and C,

1) (i) $A\ \text{U}\ B = B\ \text{U}\ A$ and
   (ii) $A \cap B = B \cap A$ (Commutativity),
2) (i) $A\ \text{U}\ (B\ \text{U}\ C) = (A\ \text{U}\ B)\ \text{U}\ C$ and
   (ii) $A \cap (B \cap C) = (A \cap B) \cap C$ (Associativity),
3) (i) $A\ \text{U}\ (B \cap C) = (A\ \text{U}\ B) \cap (A\ \text{U}\ C)$ and
   (ii) $A \cap (B\ \text{U}\ C) = (A \cap B)\ \text{U}\ (A \cap C)$ (Distributivity),
4) (i) $A\ \text{U}\ A = A$ and
   (ii) $A \cap A = A$ (Idempotence),
5) $(A^C)^C = A$ (Involution),
6) $A \subseteq B$ and $A \subseteq B \Rightarrow A \subseteq C$ (Transitivity),
7) (i) $A \cap \Omega = A$,
   (ii) $A \cap \varphi = \varphi$,
   (iii) $A\ \text{U}\ \Omega = \Omega$,
   (iv) $A \cap \Omega = A$ (Identity),
8) (i) $(A\ \text{U}\ B)^C = A^C \cap B^C$,
   (ii) $(A \cap B)^C = A^C\ \text{U}\ B^C$ (De Morgan's theorems),
9) $A\ \text{U}\ A^C = \Omega$ (Exclusion), and
10) $A \cap A^C = \varphi$ (Contradiction).

Thus the last two properties which are not satisfied if we define the complement of a fuzzy set in the way that has been in use since the beginning, are satisfied if we define the complement of a fuzzy set in our manner. Observe that properties 5 and 8 are valid with the earlier definition too. However, had the definition been correct, we should not have been able to cite a counterexample. Therefore validity in the case of these two properties is not reason enough to accept that the definition is correct. On the other hand, the invalidity in the case of the counterexamples is reason enough to discard the current definition of complementation. The counterexamples do not effect our definition, because here fuzzy membership function and fuzzy membership value have been taken to be different in the case of complement of a fuzzy set. We have gone for logic first, and then mathematics followed that logic.

Accordingly, if a normal fuzzy number $N = [\alpha, \beta, \gamma]$ is defined with a membership function $\mu_N\ (x)$, where $\mu_N(x) = \Psi_1(x)$, if $\alpha \le x \le \beta$, $= \Psi_2(x)$, if $\beta \le x \le \gamma$, and $= 0$, otherwise, while $\Psi_1\ (\alpha)$

$= \Psi_2 (\gamma) = 0$, $\Psi_1 (\beta) = \Psi_2 (\beta) = 1$, the complement N^C will have the membership function $\mu_N{}^C$ (x), where

$$\mu_N{}^C (x) = 1, \ -\infty < x < \infty,$$

with the condition that $\mu_N{}^C (x)$ is to be counted from $\Psi_1(x)$, if $\alpha \leq x \leq \beta$, from $\Psi_2(x)$, if $\beta \leq x \leq \gamma$, and from 0, otherwise, so that we keep a difference between the fuzzy membership function and the fuzzy membership value. Our definition of the complement of a fuzzy set thus is based on the following axiom:

Axiom-1: The fuzzy membership function of the complement of a normal fuzzy number N is equal to 1 for the entire real line, with the membership value counted from the membership function of N.

A blunder remains a blunder even if it continues to be used repeatedly for nearly half a century. We have two options now; either we continue using the Zadehian definition of complement of a fuzzy set, and allow the mathematics of fuzziness to be dubbed a great hoax in course of time, or we start to see reason that the counterexamples cited by us are enough to discard the old definition of complementation and rebuild the mathematics of fuzziness from level zero. We would like to repeat that our definition is necessary for defining the complement only of a normal fuzzy set. For a usual fuzzy set, we do not need this definition. Even after seeing the counterexamples that we have mentioned here, if the current workers continue using the earlier definition of the complement of a fuzzy set, we leave it to the new generation of workers to decide the fate of the mathematics of fuzziness. When a proof faces a challenge from a counterexample, the counterexample always wins. That the earlier definition is not acceptable is clear from our counterexamples. Accordingly, all mathematical formalisms and applications in which this definition of complementation had been used must be discarded forthwith. If our definition is accepted, the field theoretic matters with reference to fuzzy sets would be able to grow and to develop just like every other field of earthly mathematics.

It may be noted that the axiom defining the operation of complementation of a fuzzy set was considered unsatisfactory by quite a few earlier workers too (see for example [21]). However, the snag regarding forming a field of fuzzy sets remained. From our standpoint, the fuzzy sets do conform to the definition of a field, and therefore the field theoretic matters related to fuzzy sets should now be revised thoroughly. At this point, we would like to state that with all humility we actually called ours an extended definition of fuzziness [20]. Our definition is not an extension; it is indeed *the* definition describing any normal fuzzy set. The Zadehian definition fails to describe the complement of a fuzzy set correctly. Instead of trying to understand that the definition of complementation was defective, some people went further forward to build up a theory in which logic has been forced to follow mathematics. What has resulted is not mathematics at least, whatever else that may be.

Regarding defining axioms, we would like to add the following comments. An axiom can not just fall from the blue. One has to observe first that a particular statement is true. To define something thereafter, one may use that truth as an axiom. It is not however always necessary that what has been observed must be true. Once upon a time, people believed that the Universe was geocentric. Copernicus scrutinized the matter differently, and found that the solar system was heliocentric after all. In the same way, mathematicians once observed that the real numbers follow certain properties like sum of two real numbers is again a real number, etc. That gave the idea that a structure named group could be defined following certain axioms. That is how mathematics has been growing, and would continue to grow. One

should not just state an axiom based on plain *belief* and start a branch of study based on that kind of an axiom. That way, one might end up defining a geocentric Universe all over again!

As far as the *classical* theory of probability was concerned, we can find in the literature the statement and the proof (see for example, [22], page 27 – 28) of the Theorem of Total Probability. This theorem however was later on taken as an axiom, known as the additivity axiom, in the measure theoretic definition of probability. Here then is an example of how a theorem that could be proved while remaining in a particular system became an axiom in another system, and therefore became unprovable, following Gödelian logic, as long as we remain within the new system.

We cite another example of how a definition, mathematical or otherwise, naturally comes up. Indeed, the measure of an interval A in the real line is given by $\psi(A) = \inf \sum l(I_n)$ where the infimum is taken over all countable collections of intervals I_n of length $l(I_n)$ such that A is an improper subset of $\cup I_n$, and the intervals I_n are of the form $[a, b)$ with finite a and b . If A is not a subset of $\cup I_n$, the intervals would not entirely cover A, and so that there is no overlapping of intervals, we must take the infimum of the sum of the lengths as the measure of the set. In this case, these two conditions therefore were necessary to arrive at the required definition of a measure.

Sometimes, an axiom may even reappear possibly in a different shape and may even get a different name. Euclid's fifth axiom better known as Playfair's axiom is one such example. A straight line in a Euclidean space can be parallel to at most one of two intersecting straight lines. As far as Euclidean Geometry is concerned, here was an uprovable truth that can be actually *seen* to be true.

Using ring theoretic axioms applied to the real numbers with reference to the operations of addition and multiplication, it can be proved that $(-1).(-1) = 1$, which leads to the conclusion that the square of a real number can not be negative, and therefore that the sum of squares of real numbers can not be negative. This can be stated also as follows: given a vector a in a linear space, the scalar product $a.a$ is never negative. The readers may note that this is precisely the first of the four axioms needed to define a Euclidean space from a linear space. Here then is an example of a theorem in ring theory leading to an axiom in defining a Euclidean space.

This is how an axiom should emerge, logically and naturally, as a statement about a truth, sometimes to define a form, and sometimes to form a definition. Our logic is that when a glass is partially full of water, then the height of empty portion is to be counted from the height upto which the glass is partially full. What we are proposing can therefore be clearly *seen* from this particular example, just as Playfair's axiom can actually be *seen* to be true. Indeed, the definition of complementation of a fuzzy set should have been such that the axioms of exclusion and contradiction are satisfied. Without thinking that membership value and membership function need *not* always be the same, a definition of complementation was forwarded, which had led to invalidity of these two axioms. Instead of thinking that rather unnatural kind of a situation had been arrived at, the workers in fuzziness just followed the leader blindly, and contributed to build up a baseless theory. Acceptance of our views notwithstanding, what we are defining is the truth concerned. Even if workers from the new generation come forward to accept what we have just described, they need to understand that it might actually be very difficult for them to make most of the current workers in fuzziness see reason. Galileo was punished for supporting the Copernican views, and as we all know, history repeats itself.

4. Fuzzy randomness from our perspective

Fuzzy randomness in terms of uncertain probabilities has been studied by Buckley and Eslami ([23], [24]) and Buckley [25], among others. Assume that X is a random variable following the normal probability law with mean μ and variance unity. Now if the parameter μ is fuzzy, with membership defined in $[\mu - \delta, \mu, \mu + \delta]$, we would actually define an infinite number of normal probability density functions with location parameter ranging from $(\mu - \delta)$ to $(\mu + \delta)$ with maximum membership assigned at the value μ. This is where the current definition of fuzzy randomness ends.

Assume now that we have a normally distributed population with mean μ and variance σ^2. From this population, a sample of n observations $x_1, x_2 \ldots x_n$ has been drawn, and we can then proceed to infer about the population, based on the sample data. Assume that we have fuzzy data and we need to proceed for statistical analysis with reference to fuzzy randomness.

The fuzzified data are in terms of fuzzy numbers around x_i, $i = 1, 2, \ldots, n$ defined as, say, $X_i = [x_i - \delta, x_i, x_i + \delta]$, $\delta \geq 0$. The analysis can now proceed accordingly towards making a fuzzy statistical conclusion. Without loss of generality, and for computational simplicity, such fuzzy numbers are usually taken as triangular.

It can bee seen that equivalence of the definitions of the Dubois – Prade left reference function $\Psi_1(x)$, for $a \leq x \leq b$, and a distribution function, gives us

$$d\ \Psi_1(x)/\ dx = \varphi_1(x), \text{ say,}$$

where

$$\int_a^b \varphi_1(x)\ dx = 1.$$

Similarly, equivalence of the definitions of the Dubois – Prade right reference function $\Psi_2(x)$, for $b \leq x \leq c$, and a complementary distribution function, gives us

$$d\ (1 - \Psi_2(x))\ /\ dx = \varphi_2(x), \text{ say,}$$

where

$$\int_b^c \varphi_2(x)\ dx = 1.$$

Now according to our analysis, a triangular fuzzy number of the type $X_i = [x_i - \delta, x_i, x_i + \delta]$ with membership function

$$\mu(x) = (x - x_i + \delta)/\ \delta, \text{ if } x_i - \delta \leq x \leq x_i,$$
$$= (x_i + \delta - x)/\ \delta, \text{ if } x_i \leq x \leq x_i + \delta,$$
$$= 0, \text{ otherwise,}$$

is in fact defined by two laws of randomness with distribution functions

$$F_1(x) = (x - x_i + \delta)/\ \delta, \text{ if } x_i - \delta \leq x \leq x_i,$$

and

$$F_2(x) = 1 - (x_i + \delta - x)/\ \delta, \text{ if } x_i \leq x \leq x_i + \delta,$$

so that their densities

$$dF_1(x)/\ dx = 1/\ \delta, \text{ if } x_i - \delta \leq x \leq x_i,$$

and

$$dF_2(x)/\ dx = 1\ /\ \delta, \text{ if } x_i \leq x \leq x_i + \delta$$

are uniform.

This is how the question of fuzzy randomness should come up. First, there should be a variable following some law of randomness. Secondly, around every realization of the random variable, there should be fuzziness. The conclusions arrived at from analysis of such data will also be in terms of fuzziness. The present author had actually used the concept of fuzzy randomness quite a few years earlier to the works cited above ([23], [24], [25]) in analyzing fuzzy data available in the form of a Latin Square Layout of experimental data [26] taking care however of not mentioning the need of using two laws of randomness to define a normal fuzzy law.

We have mentioned here how exactly such a situation can come up. What we now have to add is that every fuzzy observation in a sample would be governed by two laws of randomness already. In addition, there would be a probability law that is supposed to be followed by the random variable which we have fuzzified to get an approximate reasoning. In other words, we have to add the following explanations. As soon as we surmise that the data are fuzzy, or in other words that the data are of the interval type with appropriately defined membership functions, we presume that there is one probability law of in $[\mu - \delta, \mu]$ while there is another probability law of in $[\mu, \mu + \delta]$. The readers would note here that for probabilistic conclusions based on fuzzy random data, we need to define two probability laws in the statistical sense in the two concerned intervals $[\mu - \delta, \mu]$ and $[\mu, \mu + \delta]$. In fact, if we say that the fuzzy membership function defined on $[\mu - \delta, \mu, \mu + \delta]$ is given by $\xi(x) = \Psi_1(x)$, if $\mu - \delta \le x \le \mu$, $= \Psi_2(x)$, if $\mu \le x \le \mu + \delta$, and $= 0$, otherwise, then the probability that the probability density function of the random variable X would be

$$f(x, \mu^*) = exp\ \{-(x - \mu^*)^2\}\ /\ \surd\ (2\pi),\ -\infty \le x < \infty,$$

is $\Psi_1(\mu^*)$, if $\mu - \delta \le \mu^* \le \mu$, and is $(1 - \Psi_2(\mu^*))$, if $\mu \le \mu^* \le \mu + \delta$. So in a fuzzy random case, we are able to describe the situation in terms of the probability that the underlying probability density function would be of a particular format, and only then the mathematical explanation of fuzzy randomness would actually be complete. The question of drawing inferences from such fuzzy data now remains to be discussed in the following way.

We first cite a numerical example of what we are trying to express. In our work cited earlier [26], we started with data of the interval type $[x - 0.5, x, x + 0.5]$ with an assumption that the data are triangular. The random variable X of which x is a realization in the sample was assumed to be normally distributed. In other words, we started with an assumption that the two uniform probability laws, one on $[x - 0.5, x]$ and the other on $[x, x + 0.5]$, are uniform, for a normally distributed realization x with mean μ and error variance σ^2, say. We finally arrived at a fuzzy value of Snedecor's F - statistic with the following fuzzy membership function:

$$\mu_F(x) = 174.303\ x\ /\ (157.733\ x + 17.992),\ if\ 0 \le x \le 0.482,$$
$$= 86.858\ /\ (78.866 + 16.570\ x),\ if\ 0.482 \le x < \infty.$$

Accordingly, from our standpoint this fuzzy number had been decided by the following two distribution functions:

$$\Psi_1(x) = 174.303\ x\ /\ (157.733\ x + 17.992),\ if\ 0 \le x \le 0.482,$$

and

$$(1 - \Psi_2(x)) = 1 - \{86.858\ /\ (78.866 + 16.570\ x)\},\ if\ 0.482 \le x < \infty.$$

This means, the fuzzy number $F = [0, 0.482, \infty)$ with left and right reference functions $\Psi_1(x)$ and $\Psi_2(x)$ defined in $0 \le x \le 0.482$ and $0.482 \le x < \infty$ respectively, would be defined by the two densities $d\ (\Psi_1\ (F))/\ dF$ and $d\ (1 - \Psi_2\ (F))/\ dF$ in the respective ranges.

In the non-fuzzy situation as described in [26], we would have concluded that there is no reason to reject the null hypothesis of equality of the treatment means at 5% probability level of significance as the data dependent value of $F\ (= 0.482)$ is smaller than the theoretical value of $F\ (= 4.7571)$, (see for example [27], page 536) for 3 and 6 degrees of freedom. We now proceed to look into the matters of making a fuzzy conclusion statistically. The theoretical non-fuzzy value of $F\ (= 4.7571)$ to the right of which the area under the probability density function of Snedecor's F is 0.05, is on that part of the interval on which the right reference function is defined. We note that the membership for $F = 4.7571$ is $\Psi_2\ (4.7571)$. Now, in our perspective the probability density function concerned would be given by

$$d\ (1 - \Psi_2\ (F))/\ dF = 1439.23706/\ (78.866 + 16.570\ F)^2,\ for\ 0.482 \le F < \infty.$$

Therefore the probability that $F \geq 4.7571$ would be the area under this probability density function for $F \geq 4.7571$, which is the area of the right tail beyond 4.5751. The area of the left tail from 0.482 to 4.5751 is $(1 - \Psi_2 (4.7571))$. Thus the area of the right tail is $\Psi_2 (4.7571)$ again, which is nothing but the membership value of F at 4.7571. $\Psi_2 (4.7571) = 0.5508$ is therefore the probability that the fuzzy null hypothesis of equality of treatment means would have to be rejected at 5% probability level of significance. In other words, when a non-rejectable hypothesis is fuzzified, there will still be a probability that the fuzzy hypothesis would actually be found rejectable. In the same way, if a rejectable hypothesis is fuzzified, there would still be a probability that the fuzzy hypothesis would be found non-rejectable, the probability of non-rejection being decided by the left reference function this time.

Fuzzy randomness defined in our way can be of use in sampling theory and therefore in demography. In sampling, for example, the age of a person is usually taken as an integer. In fact, that should not be the case. A better way to deal with this would be to consider the data as intervals of length unity, and then assume that the data are in terms of triangular fuzzy numbers. As soon as we assume without loss of generality that the data are triangular fuzzy numbers, we would assume uniform laws of randomness to the left and to the right of the points of maximum membership. That way, the analysis would be more practical.

With reference to fuzzy randomness, we would like to add one particular comment. In testing the rejectability of a fuzzy hypothesis, the workers have always referred to the alternative hypothesis as the complement of the fuzzy set defined in the current manner. According to us, such a definition of complementation is not correct. Therefore in every case of fuzzy statistical hypothesis testing available in the literature, the alternative hypotheses must necessarily be redefined.

We would like to add one further comment in this regard. If we discard the old view and look into the matters from our standpoint of defining normal fuzziness with reference to two probability measures, then fuzzy randomness can have one more explanation. In the case of the classical Bayesian statistical mathematics, a prior probability law is assumed with reference to a parameter associated with the law of randomness of the variable concerned. Here in our explanations, we presume two laws of randomness defining fuzziness of a parameter associated with a random variable following some probability law of errors. In fact, not all in the Statistics fraternity give credence to the Bayesian statistical analysis. But fuzziness is not just an abstract concept. Hence, instead of presuming *one* prior probability law followed by the parameter concerned defining a law of randomness, we may presume a law of fuzziness followed by the parameter. That law of fuzziness in turn will be explainable in terms of two laws of randomness. That way, instead of Bayesian analysis, we may opt for an analysis with reference to fuzzy randomness, and arrive at a conclusion based on approximate reasoning.

5. Randomness as a special case of fuzziness from our perspective

There has been a lot of unnecessary confusion regarding whether a law of fuzziness defined over an interval can in some way be related to a law of randomness defined in the same interval. We are now going to describe the one and only situation in which that is possible. In no other way can we infer a law of randomness from a law of fuzziness or a law of fuzziness from a law of randomness.

An immediate consequence of Theorem-1 stated in Section-2 is the following corollary ([28], page 51). Can a normal law of fuzziness lead to *one* law of randomness? The answer is,

yes, and it can be attained in the following manner only. This result too, just as the others included in this article, is based on plain logic, and not on unreasonable belief. If theorem-1 is true, then the following statement has to be equally true.

Corollary 1: For a fuzzy number $N = [\alpha, \beta, \beta]$ with membership function
$$\mu_N(x) = \Psi(x), \text{ if } \alpha \leq x \leq \beta,$$
$$= 0, \text{ otherwise,}$$
where $\Psi(\alpha) = 0$ and $\Psi(\beta) = 1$, with nondecreasing $\Psi(x)$ in $\alpha \leq x \leq \beta$, $\Psi(x)$ has to be a probability distribution function in the interval $[\alpha, \beta]$.

In other words, every interval that is a probability space is basically a fuzzy number. For example, if we have an exponentially distributed random variable X with the probability density function
$$f(x) = exp\{-x/\sigma\}/\sigma, 0 \leq x < \infty,$$
its probability distribution function will be given by
$$F(x) = 1 - exp\{-x/\sigma\}, 0 \leq x < \infty.$$
In our way, $[0, \infty)$ then is a fuzzy number with membership function $F(x)$.

In fact, if X is a random variable following a probability law in the interval $[\alpha, \beta]$, then we shall have for $\alpha \leq \gamma \leq \delta \leq \beta$,
$$\text{Prob}[\gamma \leq X \leq \delta] = \text{Prob}[\alpha \leq X \leq \delta] - \text{Prob}[\alpha \leq X \leq \gamma]$$
$$= \mu(\delta) - \mu(\gamma),$$
where $\mu(\gamma)$ and $\mu(\delta)$ are the membership values at γ and δ respectively.

It is clear that from our standpoint theory of probability could be taken as a special case of the theory of normal fuzziness. To deduce the probability law of a function of a random variable, for example to find the probability law of X^2 when the probability law of X is known, the usual statistical procedure is to deduce the probability density of the transformed variable based on the probability law of the given variable. If we use our method, the membership function of the new variable can be found from the membership function of the given variable, and after differentiation of the membership function we would get the probability density function of the transformed variable. Matters related to distribution theory may therefore be studied in this way too. This may actually lead to simplifying certain statistical formalisms existing in the literature.

We would like to cite an example. Tshebysheff's Lemma ([22], page 182) is stated as follows: Let u be a variable which does not assume negative values, and a be its mathematical expectation. Then
$$\text{Prob}\{u \leq at^2\} \geq 1 - (1/t^2)$$
whatever t may be. This is a very general statement in the sense that it is independent of the underlying probability law followed by the variable. In our perspective, this lemma can equivalently be stated as follows: Let $U = [0, \infty)$ be a fuzzy number with nondecreasing membership function $\mu(u)$, with $\mu(0) = 0$ and $\mu(u)$ tending to 1 as u tends to ∞. Then

$$\mu(at^2) \geq 1 - 1/t^2$$
where
$$a = \int_0^\infty u \, d\mu(u).$$

This is the only way to get one law of randomness from one law of fuzziness, and to get one law of fuzziness from one law of randomness. There is no other alternative.

We have come across situations where fuzziness for a certain parameter and statistical randomness for another, both occurring in the same mathematical model, have been suggested. In our sense, both of such parameters are fuzzy with the original fuzzy parameter defined as a fuzzy number of the type [α, β, γ] and with the probabilistic parameter defined as a fuzzy number of the type [α, β, β]. In other words, what is known as a mixed model is indeed a model with two different types of fuzziness, one with both left as well as right reference functions, while the other is with just the left reference function.

6. Conclusions

The following points are to be included in the theory of fuzzy sets so as to make it truly logical. Partial presence of an element in a normal fuzzy number can in fact be expressed in terms of randomness. The membership function of a normal fuzzy number [α, β, γ] is actually a distribution function in [α, β] and a complementary distribution function in [β, γ]. Therefore trying to conclude anything in terms of the area under the membership function is absolutely meaningless because integration of both a distribution function and a complementary distribution function does not have any physical significance. Accordingly, trying to bring in some sort of equivalence between the area under the membership function and the area under a probability density function defined in the same interval is not logical because an area under the membership function can not in any way lead to any kind of probability, while an area under a probability density function is the probability of an event. Hence, the existing probability – possibility consistency principles in turn are not logical. In fact, if we look from our perspective, *measures* such as belief, plausibility, credibility etc. are not at all mathematical in the classical sense.

The definition of complement of a fuzzy set is *not* correct. Hence all sorts of mathematical and applicational matters in which this definition had been used must also be wrong. Indeed, fuzzy membership function and fuzzy membership value are two different things with reference to the complement of a normal fuzzy set. The membership function of the complement of a normal fuzzy number is 1 over the entire real line, with the condition that it has to be counted from the membership function of the original fuzzy number. To continue applying a wrong axiom just because it has been arrogantly believed to true for nearly half a century goes totally against the philosophy of mathematics. Meanwhile this unearthly definition has been used so extensively in all sorts of fields for so many years that it may now be actually impossible to undo most of the things done so far in this context.

Fuzzy randomness leads to fuzzy conclusions. Such fuzzy conclusions can indeed be viewed probabilistically also. Note that here too in testing of fuzzy hypotheses the definition of alternative hypotheses concerned were all wrongly stated. Further, a fuzzy random situation may be viewed as an extension of the Bayesian statistical analytical procedure if we look into the matters defining fuzziness from our perspective.

Finally, the theory of probability can be studied as a special case of the theory of fuzziness with reference to fuzzy numbers of the type [α, β, β]. This in itself looks a vast field of study.

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Author

Hemanta Kumar Baruah, Professor, Department of Statistics, and Former Dean, Faculty of Science, Gauhati University, is a Reviewer in the Mathematical Reviews, USA. He received his M. Sc. degree in Statistics from Gauhati University in 1975, and his Ph. D. degree in Mathematics from Indian Institute of Technology, Kharagpur, in 1980. His current research interests are in mathematics of fuzziness, graph theory, pattern recognition, data mining, neural network analysis and mathematical modeling.

This work has been dedicated in honor of my revered teacher Dr. Prantosh Dhar, Retired Professor, Department of Electronics and Electrical Communications Engineering, Indian Institute of Technology, Kharagpur. It was Professor Dhar who had instigated me way back in 1976 to do some bit of original research. – Author.