Towards Forming A Field Of Fuzzy Sets

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Abstract

It has been accepted that the fuzzy sets do not form a field. In this article, we are going to put forward an extension of the definition of fuzziness. With the help of this extension, we would be able to define the complement of a fuzzy set properly. This in turn would allow us to assert that fuzzy sets do form a field. In fact, the fuzzy membership value and the fuzzy membership function for the complement of a fuzzy set are two different things. This confusion has created a stumbling block towards accepting the theory of fuzzy sets as a generalization of the classical theory of sets.

Keywords: Complement of a fuzzy set, fuzzy membership value, fuzzy membership function.

1. Introduction

The theory of fuzzy sets should actually have been a generalization of the classical theory of sets in the sense that the theory of sets should have been a special case of the theory of fuzzy sets. Unfortunately, this is not the case. It has been accepted that for a fuzzy set A and its complement A^C, neither $A \cap A^C$ is the null set, nor $A \cup A^C$ is the universal set. Accordingly, the fuzzy sets have been accepted not to form a field (see e.g. [1], pp-34). Whereas the operations of union and intersection of two crisp sets are indeed special cases of the corresponding operations of two fuzzy sets, they end up giving peculiar results while defining $A \cap A^C$ as well as $A \cup A^C$. An obvious question should have arisen in this context much earlier. Has the operation of complementation of a fuzzy set been defined correctly? Obviating this seemingly natural question, and therefore accepting that the fuzzy sets do not form a field, a large number of articles and books have been published worldwide. Our standpoint in this regard is rather straightforward. Mathematics must necessarily be acceptable as absolute truth, and in search of truth we may sometimes have to throw out illogical matters. We would like to start with the perspective that the manner in which complementation of a fuzzy set is defined is *wrong*, which is why the graphical representations of $A \cap A^C$ and $A \cup A^C$ lead to peculiar results. This in turn has given rise to all sorts of peculiar mathematical formalisms which the mathematicians working outside the realm of fuzzy mathematics find very hard to believe.

In this article, we shall put forward an extended definition of fuzzy sets ([2], pp – 9-17). This would enable us to define the complement of a fuzzy set in a way that gives us $A \cap A^C$ = the null set φ, and $A \cup A^C$ = the universal set Ω. This in turn would be sufficient to state that fuzzy sets indeed form a field. This extended definition would however be required only to define the complement of a fuzzy set properly. For all other matters, this extended definition is not needed. Our only objective is to assert that the definition of complementation in vogue is incorrect. We are going to propose the correction now. We know that what we are claiming is precisely opposite of what has been accepted to be true for the last forty years or so. But a truth must absolutely be based on logic, and not on popular belief. We must not keep on repeating a mistake just because a lot has been published worldwide based on that one mistake.

2. An extended definition of fuzzy sets

Not everything can be counted from the zero level. For example, say at a particular place a mineral is available from a depth of 50 meters to a depth of 200 meters. Indeed, at that place the actual depth of the mineral bed is 150 meters, but it is to be counted from a depth of 50 meters. In other words, the depth of a mineral bed cannot be counted from the zero level which in this case is the surface of the Earth. We start with this simple reasoning to extend the definition of fuzzy sets.

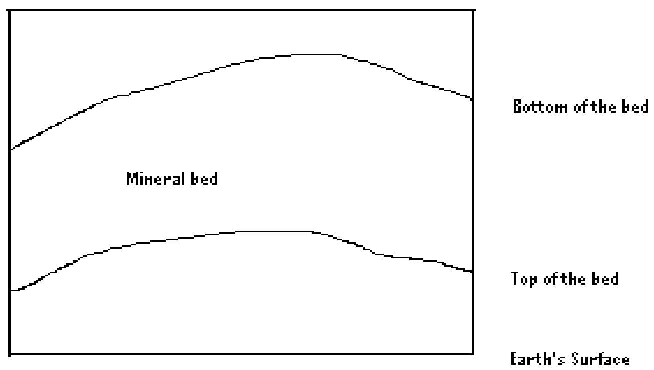

Figure 1 An Upside Down View of a Mineral Bed

Consider an upside down view of the aforesaid mineral bed (Figure 1). Say, for a given x measured from some point of reference on the surface of the Earth, the mineral is available upto a depth , or height in the upside down view, $h_1(x)$ to a depth, or height in the upside down view, $h_2(x)$ so that thickness of the mineral bed is $(h_2(x) - h_1(x))$. One can see that $h_2(x)$ is like a membership function measured from a function of reference $h_1(x)$ such that $(h_2(x) - h_1(x))$ can be said to be the actual value of *membership*.

We now proceed to use this simple idea to extend the definition of fuzzy numbers. It is known that a fuzzy number [a, b, c] is defined with reference to a membership function $\mu(x)$ lying between 0 and 1, $a \leq x \leq c$. We would like to extend this definition in the following way. Let $\mu_1(x)$ and $\mu_2(x)$ be two functions, $0 \leq \mu_2(x) \leq \mu_1(x) \leq 1$. We would call $\mu_1(x)$ the *fuzzy membership function*, and $\mu_2(x)$ a *reference function*, such that $(\mu_1(x) - \mu_2(x))$ is the *fuzzy membership value* for any x. we would like to characterize such a fuzzy number by $\{x, \mu_1(x), \mu_2(x); x \in \Omega\}$.

We next proceed to define a set operation named *superimposition*. When we overwrite, the overwritten portion looks darker. The operation of union of sets cannot explain this. Defined by the present author ([3], [4]), and used successfully in recognizing periodic patterns [5], the operation of set superimposition is defined as follows: if the set A is *superimposed* over the set B, we get

$$A\,(S)\,B = (A-B) \cup (A \cap B)^{(2)} \cup (B-A)$$

where S represents the operation of superimposition, and $(A \cap B)^{(2)}$ represents the elements of $(A \cap B)$ occurring twice.

Assume now that $A^{(\mu)}$ and $A^{(\upsilon)}$ are two fuzzy sets over the same support with membership $\mu(x)$ and $\upsilon(x)$ respectively for all concerned x. For simplicity, assume that $\mu(x) + \upsilon(x) \leq 1$. If we now superimpose $A^{(\upsilon)}$ over $A^{(\mu)}$, according to our characterization principle, $A^{(\mu)}$ would be characterized by $\{x, \mu(x), 0; x \in \Omega\}$ while the superimposed $A^{(\upsilon)}$ would be characterized by $\{x, \mu(x) + \upsilon(x), \mu(x); x \in \Omega\}$. It is obvious that $\{x, \mu(x) + \upsilon(x), 0; x \in \Omega\}$ would then characterize the fuzzy set $A^{(\mu+\upsilon)}$ obtained due to the superimposition of $A^{(\upsilon)}$ over $A^{(\mu)}$. This is something like placing one translucent paper over another translucent paper of equal opacity to get the opacity doubled as a result.

In what follows, we shall see how our extended definition works towards finding union and intersection of fuzzy sets. This will in turn help us to define complementation of a fuzzy set in its true perspective. If our definition of complement of a fuzzy set is found acceptable, it will finally lead to the assertion that the theory of fuzzy sets is indeed a generalization of the classical theory of sets.

3. Application of the extended definition

We now proceed to see what happens to $A \cap B$ and $A \cup B$ when we extend the definition of fuzziness in our way. Say,

$$A(\mu_1, \mu_2) = \{x, \mu_1(x), \mu_2(x); x \in \Omega\}$$

and

$$B(\mu_3, \mu_4) = \{x, \mu_3(x), \mu_4(x); x \in \Omega\}.$$

Then we would have

$$A(\mu_1, \mu_2) \cap B(\mu_3, \mu_4) = \{x, \min(\mu_1(x), \mu_3(x)), \max(\mu_2(x), \mu_4(x)); x \in \Omega\}$$

and

$$A(\mu_1, \mu_2) \cup B(\mu_3, \mu_4) = \{x, \max(\mu_1(x), \mu_3(x)), \min(\mu_2(x), \mu_4(x)); x \in \Omega\}.$$

Two fuzzy sets $C = \{x, \mu_C(x); x \in \Omega\}$ and $D = \{x, \mu_D(x); x \in \Omega\}$ in the usual definition would be expressed as $C(\mu_C, 0) = \{x, \mu_C(x), 0; x \in \Omega\}$ and $D(\mu_D, 0) = \{x, \mu_D(x), 0; x \in \Omega\}$ in our way. Accordingly, we would have

$$\begin{aligned} C(\mu_C, 0) \cap D(\mu_D, 0) &= \{x, \min(\mu_C(x), \mu_D(x)), \max(0, 0); x \in \Omega\} \\ &= \{x, \mu_C(x) \wedge \mu_D(x); x \in \Omega\} \end{aligned}$$

which in the usual definition is nothing but $C \cap D$.

Similarly,

$$\begin{aligned} C(\mu_C, 0) \cup D(\mu_D, 0) &= \{x, \max(\mu_C(x), \mu_D(x)), \min(0, 0); x \in \Omega\} \\ &= \{x, \mu_C(x) \vee \mu_D(x); x \in \Omega\} \end{aligned}$$

which in the usual definition is nothing but $C \cup D$.

Thus we have seen that for union and intersection of two fuzzy sets, the extended definition leads to the union and intersection under the standard definition. Let us now look into what happens to the operation of complementation from our standpoint.

4. Complement of a fuzzy set using the extended definition

Consider first the usual definition of a fuzzy number. Let A be a fuzzy number characterized by $A = \{x, \mu(x); x \in \Omega\}$. Its complement A^C is characterized by (see e.g. [6])

$$A^C = \{x, (1 - \mu(x)); x \in \Omega\}.$$

The diagram concerned looks as the one shown in Figure 2. Now A and A^C looks like having something common, which is why it has been accepted that $A \cap A^C \neq \varphi$. Further, as looks obvious from the diagram, $A \cup A^C \neq \Omega$. For these two inequalities, it has been accepted that the fuzzy sets do not form a field.

Now for two fuzzy sets $A(\mu, 0) = \{x, \mu(x), 0; x \in \Omega\}$ and $B(1, \mu(x)) = \{x, 1, \mu(x); x \in \Omega\}$ defined over the same universe Ω, we would have

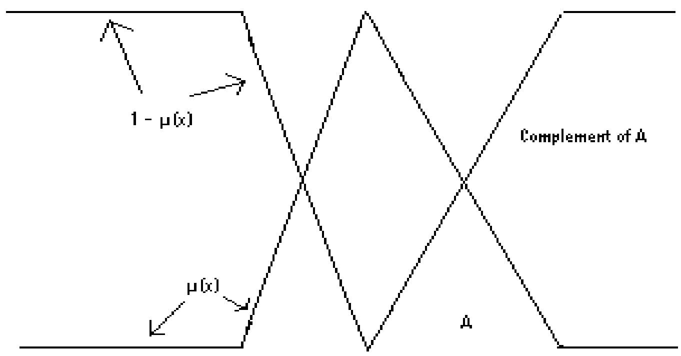

Figure 2 Complement of a Fuzzy Set: the Current Definition

$$A(\mu, 0) \cap B(1, \mu) = \{x, \min(\mu(x), 1), \max(0, \mu(x)); x \in \Omega\}$$
$$= \{x, \mu(x), \mu(x); x \in \Omega\}$$

which is nothing but the null set φ. In other words, $B(1, \mu)$ defined above is nothing but $(A(\mu, 0))^C$ in the classical sense of set theory. This means, if we define the fuzzy set

$$(A(\mu, 0))^C = \{x, 1, \mu(x); x \in \Omega\},$$

it can be seen that it should be nothing but the complement of the fuzzy set

$$A(\mu, 0) = \{x, \mu(x), 0; x \in \Omega\}.$$

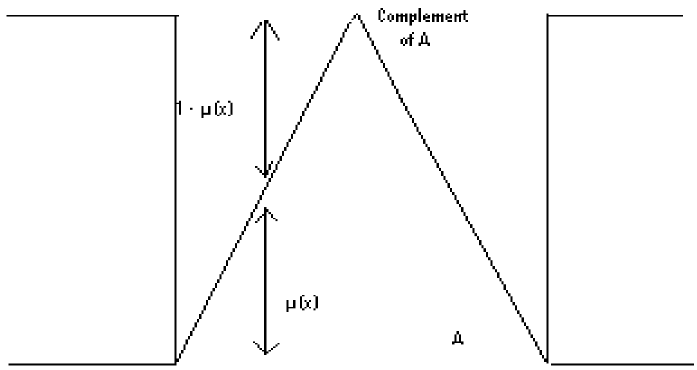

Figure 3. Complement of a Fuzzy Set: the Proposed Definition

In Figure 3, we can observe that for a fuzzy number A = [a, b, c], the value of membership for any $x \in \Omega$ is given by $\mu(x)$ for $a \leq x \leq c$, and is *zero* otherwise. For the fuzzy number A^C, the value of membership for any $x \in \Omega$ is given by $(1 - \mu(x))$ for $a \leq x \leq c$, and is 1 otherwise. The only difference is that for A^C the *membership function* is 1 everywhere with the *reference function* being $\mu(x)$, while for A the *membership function* is $\mu(x)$ with the *reference function* being 0 everywhere.

Coming to union of these two fuzzy sets, we see that

$$A(\mu_A, 0) \cup B(1, \mu_A) = \{x, \max(\mu_A(x), 1), \min(0, \mu_A(x)); x \in \Omega\}$$
$$= \{x, 1, 0; x \in \Omega\}$$

which is nothing but the universal set Ω.

We therefore conclude that if we express the complement of a fuzzy set $A = \{x, \mu_A(x), 0; x \in \Omega\}$ as $A^C = \{x, 1, \mu_A(x); x \in \Omega\}$, we get

i) $A \cap A^C$ = the null set φ, and

ii) $A \cup A^C$ = the universal set Ω.

This would enable us to establish that the fuzzy sets do form a field if we define complementation in our way.

5. Conclusions

We have seen that if a fuzzy set is characterized with respect to a reference function, we can define the complement of a fuzzy set in its actual perspective. This allows us to remove the difficulty that had debarred us to assert that contrary to what has been accepted till this day, the fuzzy sets do form a field in the classical sense. Hence, the classical theory of sets can indeed be viewed as a special case of the theory of fuzzy sets, if the complement of a fuzzy set is defined using our standpoint. *Indeed, the fuzzy membership value and the fuzzy membership function for the complement of a fuzzy set are two different things although for a usual fuzzy set they are no different because the value of the function is counted from zero in the usual case. This confusion has to be removed so as to view the theory of fuzzy sets as a generalization of the classical theory of sets.*

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