International Journal of Energy, Information and Communications
Vol. 1, Issue 1, November, 2010

THE RANDOMNESS – FUZZINESS CONSISTENCY PRINCIPLE

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AMS 2000 Classification: 03E72

Abstract

Since the beginning, for the last forty five years, it has been accepted that fuzziness and randomness are independent concepts. Workers in fuzzy mathematics have been trying to link randomness with fuzziness without any success. In this article, we would establish that trying to impose one probability space over an interval defining a possibility space is not correct. We need two probability spaces to define a possibility space. On the other hand, given a possibility space, it can be defined by two probability spaces. Hence the measure theoretic matters of possibility must be geared to forming two probability measures. The correct randomness - fuzziness consistency principle is:

$$Poss\,[x] = \theta\, Prob\,[a \le y \le x] + (1 - \theta)\, \{1 - Prob\,[b \le y \le x]\},$$

where θ is 1, if $a \le x \le b$, and is 0, if $b \le x \le c$. In other words, fuzziness is rooted at randomness.

Key words: *Probability measure*, *fuzzy membership function*.

1. Introduction

The theory of fuzzy sets came into existence forty five years ago as a competitor of the theory of probability to describe uncertainties that are not describable using probability measures. A normal fuzzy number N = [α, β, γ] is associated with a membership function $\mu_N(x)$, where

$$\begin{aligned}\mu_N(x) &= \Psi_1(x), \text{ if } \alpha \le x \le \beta,\\ &= \Psi_2(x), \text{ if } \beta \le x \le \gamma, \text{ and}\\ &= 0, \text{ otherwise,}\end{aligned}$$

$\Psi_1(x)$ being a continuous *nondecreasing* function in the interval [α, β], and $\Psi_2(x)$ being a continuous *nonincreasing* function in the interval [β, γ], with $\Psi_1(\alpha) = \Psi_2(\gamma) = 0$, $\Psi_1(\beta) = \Psi_2(\beta) = 1$. Dubois and Prade (see for example, [1]) named $\Psi_1(x)$ as the *left reference function* and $\Psi_2(x)$ as the *right reference function* of the fuzzy number N = [α, β, γ].

Dubois and Prade however stopped short of defining wherefrom these two functions come up. We are interested in viewing $\Psi_1(x)$ and $(1 - \Psi_2(x))$ as probability distribution functions and would study the membership function of a fuzzy number from this angle. We would first look into how such distribution functions can be constructed so that every fuzzy number can be explained with the help of *two* probability measures. For this we shall first define a set operation called superimposition, and use a classical result of Order Statistics on uniform convergence of empirical distribution functions thereafter to arrive at our result.

The Dubois – Prade definition was a firm step towards defining fuzziness with the help two *different* functions. We would like to start where they have stopped. To go further, we would need the spectacles of superimposition of sets which is our own finding. Further, we would need to apply the Glivenko – Cantelli theorem of order statistics to arrive at our conclusions.[1]

2. The operation of set superimposition

We first proceed to define a set operation that we have named *superimposition*. When we overwrite, the overwritten portion looks darker. Indeed, in the overwritten portion there happens to double representation due to *superimposition*, which is why that portion looks darker. The operation of union of sets cannot explain this. When two translucent papers with unequal opacities are placed one covering the other partially, the opacity in the portion covered by both the papers would be more than the maximum opacity in comparison with the other parts. This happens due to superimposition. We now proceed to define this mathematically.

Defined by the present author ([2], [3]) and later used successfully in recognizing periodic patterns ([4], [5], [6]), the operation of set superimposition is expressed as follows: if the set A is *superimposed* over the set B, we get

$$A\,(S)\,B = (A\text{-}B) \cup (A \cap B)^{(2)} \cup (B\text{-}A)$$

where S represents the operation of superimposition, and $(A \cap B)^{(2)}$ represents the elements of $(A \cap B)$ occurring twice, provided that $(A \cap B)$ is not void. We have defined this operation keeping view that fact that if two line segments A and B of unequal lengths are overdrawn one over the other, this is what we are going to see.

It can be seen that for two intervals $A = [a_1, b_1]$ and $B = [a_2, b_2]$, we should have equivalently

$$\begin{aligned} &[a_1, b_1]\,(S)\,[a_2, b_2] \\ &= [a_1, a_2] \cup [a_2, b_1]^{(2)} \cup [b_1, b_2], \text{ if } a_1 < a_2 < b_1 < b_2, \\ &= [a_1, a_2] \cup [a_2, b_2]^{(2)} \cup [b_2, b_1], \text{ if } a_1 < a_2 < b_2 < b_1, \\ &= [a_2, a_1] \cup [a_1, b_1]^{(2)} \cup [b_1, b_2], \text{ if } a_2 < a_1 < b_1 < b_2, \\ &= [a_2, a_1] \cup [a_1, b_2]^{(2)} \cup [b_2, b_1], \text{ if } a_2 < a_1 < b_2 < b_1, \end{aligned}$$

where $a_1 < a_2 < b_1 < b_2$, $a_1 < a_2 < b_2 < b_1$, $a_2 < a_1 < b_1 < b_2$, and $a_2 < a_1 < b_2 < b_1$ are the four different possibilities in this case. Here we have assumed without loss of any generality that

$$[a_1, b_1] \cap [a_2, b_2]$$

is not void, or in other words $\max(a_i) \leq \min(b_i)$, $i = 1, 2$.

We can express this as follows. Indeed

$$[a_1, b_1]\,(S)\,[a_2, b_2] = [a_{(1)}, a_{(2)}] \cup [a_{(2)}, b_{(1)}]^{(2)} \cup [b_{(1)}, b_{(2)}]$$

where

$$a_{(1)} = \min(a_1, a_2),$$

[1] This work was read as an invited talk in the *International Congress of Mathematics Satellite International Conference on Probability and Statistics*, September 1 – 3, 2010, Sambalpur University, Sambalpur, India.

$$a_{(2)} = \max(a_1, a_2),$$
$$b_{(1)} = \min(b_1, b_2),$$

and

$$b_{(2)} = \max(b_1, b_2).$$

This conversion in terms of ordered values is to be noted properly. We would soon see the applicability of this conversion in defining the randomness-fuzziness principle.

In this way, for n intervals $[a_1, b_1]$, $[a_2, b_2]$, ………, $[a_n, b_n]$, subject to the condition that

$$[a_1, b_1] \cap [a_2, b_2] \cap \ldots\ldots\ldots \cap [a_{n-1}, b_{n-1}] \cap [a_n, b_n]$$

is not void, we would have $(n!)^2$ different cases that can be in short written as

$$\begin{aligned}&[a_1, b_1]\ (S)\ [a_2, b_2]\ (S)\ \ldots\ldots\ldots\ (S)\ [a_{n-1}, b_{n-1}]\ (S)\ [a_n, b_n]\\ &= [a_{(1)}, a_{(2)}] \cup [a_{(2)}, a_{(3)}]^{(2)} \cup \ldots\ldots\ldots \cup [a_{(n-1)}, a_{(n)}]^{(n-1)} \cup [a_{(n)}, b_{(1)}]^{(n)}\\ &\quad \cup [b_{(1)}, b_{(2)}]^{(n-1)} \cup \ldots\ldots\ldots \cup [b_{(n-2)}, b_{(n-1)}]^{(2)} \cup [b_{(n-1)}, b_{(n)}],\end{aligned}$$

where $a_{(1)}, a_{(2)}, \ldots\ldots\ldots, a_{(n)}$ are values of $a_1, a_2, \ldots\ldots\ldots, a_n$ arranged in increasing order of magnitude, and $b_{(1)}, b_{(2)}, \ldots\ldots\ldots, b_{(n)}$ also are values of $b_1, b_2, \ldots\ldots\ldots, b_n$ arranged in increasing order of magnitude, and for example $[a_{(n-1)}, a_{(n)}]^{(n-1)}$ are elements of $[a_{(n-1)}, a_{(n)}]$ represented (n-1) times. Observe that order statistical matters can now enter into our discussions on superimposition.

3. An application of superimposition of sets

We now proceed towards an application of the operation of set superimposition. We refer to the example of two line segments A and B of unequal lengths overdrawn one over the other again. Double representation creates a doubly dark situation in the common portion. Now if the *level* of darkness in the common portion is taken to be unity, then that in the other portions would have to be partial.

For *n* fuzzy intervals $[a_1, b_1]^{(1/n)}$, $[a_2, b_2]^{(1/n)}$, …, $[a_n, b_n]^{(1/n)}$ all with membership value equal to 1/n everywhere, we shall have

$$\begin{aligned}&[a_1, b_1]^{(1/n)}\ (S)\ [a_2, b_2]^{(1/n)}\ (S)\ \ldots\ldots\ldots\ (S)\ [a_n, b_n]^{(1/n)}\\ &= [a_{(1)}, a_{(2)}]^{(1/n)} \cup [a_{(2)}, a_{(3)}]^{(2/n)} \cup \ldots\ldots\ldots \cup [a_{(n-1)}, a_{(n)}]^{((n-1)/n)} \cup [a_{(n)}, b_{(1)}]^{(1)}\\ &\quad \cup [b_{(1)}, b_{(2)}]^{((n-1)/n)} \cup \ldots\ldots\ldots \cup [b_{(n-2)}, b_{(n-1)}]^{(2/n)} \cup [b_{(n-1)}, b_{(n)}]^{(1/n)},\end{aligned}$$

where, for example, $[b_{(1)}, b_{(2)}]^{((n-1)/n)}$ represents the uniformly fuzzy interval $[b_{(1)}, b_{(2)}]$ with membership ((n-1) /n) in the entire interval, $a_{(1)}, a_{(2)}, \ldots\ldots\ldots, a_{(n)}$ being values of $a_1, a_2, \ldots\ldots\ldots, a_n$ arranged in increasing order of magnitude, and $b_{(1)}, b_{(2)}, \ldots\ldots\ldots, b_{(n)}$ being values of $b_1, b_2, \ldots\ldots\ldots, b_n$ arranged in increasing order of magnitude.

Consider now two spaces (Ω_1, A_1, Π_1) and (Ω_2, A_2, Π_2), Ω_1 and Ω_2 being real intervals $[\alpha, \beta]$ and $[\beta, \gamma]$ respectively. Let $x_1, x_2, \ldots, x_n$, and $y_1, y_2, \ldots, y_n$, be realizations in $[\alpha, \beta]$ and $[\beta, \gamma]$ respectively. So for n such equally fuzzy intervals $[x_1, y_1]^{(1/n)}$, $[x_2, y_2]^{(1/n)}$, …, $[x_n, y_n]^{(1/n)}$ all with membership value equal to 1/n everywhere, we shall have

$$\begin{aligned}&[x_1, y_1]^{(1/n)}\ (S)\ [x_2, y_2]^{(1/n)}\ (S)\ \ldots\ldots\ldots\ (S)\ [x_n, y_n]^{(1/n)}\\ &= [x_{(1)}, x_{(2)}]^{(1/n)} \cup [x_{(2)}, x_{(3)}]^{(2/n)} \cup \ldots\ldots\ldots \cup [x_{(n-1)}, x_{(n)}]^{((n-1)/n)} \cup [x_{(n)}, y_{(1)}]^{(1)}\end{aligned}$$

$$\cup [y_{(1)}, y_{(2)}]^{((n-1)/n)} \cup \ldots\ldots\ldots \cup [y_{(n-2)}, y_{(n-1)}]^{(2/n)} \cup [y_{(n-1)}, y_{(n)}]^{(1/n)}$$

where for example $[y_{(1)}, y_{(2)}]^{((n-1)/n)}$ represents the uniformly fuzzy interval $[y_{(1)}, y_{(2)}]$ with membership ((n-1) /n) in the entire interval, $x_{(1)}, x_{(2)}, \ldots\ldots\ldots, x_{(n)}$ being values of $x_1, x_2, \ldots\ldots\ldots, x_n$ arranged in increasing order of magnitude, and $y_{(1)}, y_{(2)}, \ldots\ldots\ldots, y_{(n)}$ being values of $y_1, y_2, \ldots\ldots\ldots, y_n$ arranged in increasing order of magnitude.

Recall that for the fuzzy intervals $[x_1, y_1]^{(1/n)}, [x_2, y_2]^{(1/n)}, \ldots, [x_n, y_n]^{(1/n)}$, all with uniform membership 1/n, the values of membership of the superimposed fuzzy intervals are 1/n, 2/n, …, (n-1)/n, 1, (n-1)/n, …, 2/n, and 1/n. These values of membership considered in two halves as

$$(0, 1/n, 2/n, \ldots, (n-1)/n, 1),$$

and

$$(1, (n-1)/n, \ldots, 2/n, 1/n, 0),$$

would suggest that they can define an empirical probability distribution and a *complementary* empirical distribution on $x_1, x_2, \ldots, x_n$, and $y_1, y_2, \ldots, y_n$, respectively. In other words, for realizations of the values of $x_{(1)}, x_{(2)}, \ldots\ldots\ldots, x_{(n)}$ in increasing order and of $y_{(1)}, y_{(2)}, \ldots\ldots\ldots, y_{(n)}$ again in increasing order, we can see that if we define

$$\begin{aligned} \Psi_1(x) &= 0, \text{ if } x < x_{(1)}, \\ &= (r-1)/n, \text{ if } x_{(r-1)} \leq x \leq x_{(r)}, r = 2, 3, \ldots, n, \\ &= 1, \text{ if } x \geq x_{(n)}, \\ \Psi_2(y) &= 1, \text{ if } y < y_{(1)}, \\ &= 1 - (r-1)/n, \text{ if } y_{(r-1)} \leq y \leq y_{(r)}, r = 2, 3, \ldots, n, \\ &= 0, \text{ if } y \geq y_{(n)}, \end{aligned}$$

then we are assured that

$$\Psi_1(x) \rightarrow \Pi_1 [\alpha, x], \alpha \leq x \leq \beta,$$
$$\Psi_2(y) \rightarrow 1 - \Pi_2 [\beta, y], \beta \leq y \leq \gamma.$$

We have thus seen that existence of two densities $d\Psi_1(x)/dx$ and $d\Psi_2(y)/dy$ for $\alpha \leq x \leq \beta$ and $\beta \leq y \leq \gamma$ is a *sufficient* condition to construct a fuzzy number $[\alpha, \beta, \gamma]$. We can now summarize our findings as follows: if $\Psi_1(x)$ and $(1 - \Psi_2(x))$ are two independent distribution functions defined in $[\alpha, \beta]$ and $[\beta, \gamma]$ respectively, then the membership function of a fuzzy number $N = [\alpha, \beta, \gamma]$ can be expressed as $\mu_N(x) = \Psi_1(x)$, if $\alpha \leq x \leq \beta$, $= \Psi_2(x)$, if $\beta \leq x \leq \gamma$, and $= 0$, otherwise ([7], [8]).

Thus the existence of two uniform densities, the simplest form of all densities, in the intervals $[\alpha, \beta]$ and $[\beta, \gamma]$, is *sufficient* for the construction of a triangular fuzzy number $[\alpha, \beta, \gamma]$, the simplest form of all fuzzy numbers. Other kinds of densities would be sufficient accordingly to give rise to other kinds of fuzzy numbers. We are asserting that assumption of two densities, and hence assumption of two distribution functions in $[\alpha, \beta]$ and $[\beta, \gamma]$, would give rise to a fuzzy number. It can be expected that defining a fuzzy number in this way would be helpful in explaining fuzzy arithmetic in a much simpler way.

Hence, not one but two distributions with reference to two probability measures defined on two disjoint spaces can construct a fuzzy membership function. For this however one needs to look into the matters through application of the Glivenko – Cantelli theorem of order statistics on superimposed uniformly fuzzy intervals. The distributions may be geared to the measure theoretic definition of randomness. What we mean is that the variable concerned need not be associated with any error term as in statistics. Even when the values of the variable are already ordered, the construction would still be valid. We shall discuss more about it later

The membership function of a fuzzy set defines the concerned possibility distribution ([9], [10]). It is known that if F_Ω and P_Ω are the possibility distribution and the probability distribution respectively defined on Ω, one can not infer F_Ω and P_Ω from each other [10]. Various other authors (see for example [11]) have tried to link probability with possibility, without success.

We have already answered as to why they did not succeed in linking probability with possibility. We have with the help of the set operation called superimposition and using the Glivenko-Cantelli theorem on order statistics established that fuzziness can indeed be linked with randomness. We would show that *not one* but *two* probability distributions are needed to define a possibility distribution of a *normal* fuzzy set with membership equal to one for some element of the set. In other words, we could be able to conclude that there is no logical reason why a P_Ω should be imposed on an Ω on which an F_Ω is defined. This would therefore define a principle linking randomness and fuzziness.

We have therefore arrived at our answer to the question which we have raised earlier. We have seen that two probability spaces (Ω_1, A_1, P_1) and (Ω_2, A_2, P_2) where Ω_1 and Ω_2 are the real intervals [a, b] and [b, c] respectively can define a possibility space $\Omega = \Omega_1 \cup \Omega_2$ represented by [a, b, c]. Hence there can not be any mathematically valid reason why a P_Ω should be imposed on Ω on which an F_Ω is defined. Indeed, an Ω can not be used in the same sense for a P_Ω as well as an F_Ω.

Now, if $f(x)$ is integrable in an interval $a \le x \le b$, and

$$F(x) = \int_a^x f(x)\, dx$$

then

$$d\, F(x)/\, dx = f(x).$$

Any function $F(x)$ whose derivative is equal to $f(x)$ is a primitive or an indefinite integral of $f(x)$. Thus the definite integral regarded as a function of the upper limit of the integral, is a primitive of the integral whenever the latter is a continuous function.

It can be seen that equivalence of the definitions of the Dubois – Prade left reference function $\Psi_1(x)$, if $a \le x \le b$, and a distribution function, gives us

$$d\, \Psi_1(x)/\, dx = \varphi_1(x), \text{ say,}$$

where

$$\int_a^b \varphi_1(x)\, dx = 1.$$

Similarly, equivalence of the definitions of the Dubois – Prade right reference function $\Psi_2(x)$, if $b \le x \le c$, and a complementary distribution function, gives us

$$d\, (1 - \Psi_2(x))\, /\, dx = \varphi_2(x),$$

say, where

$$\int_b^c \varphi_2(x)\, dx = 1.$$

This means, our condition is not just sufficient but *necessary* as well in the sense that a possibility space can be bifurcated into two *distinct* probability spaces. In other words, a possibility space can be defined on a triad [a, b, c] *if and only if* two probability spaces can be defined on [a, b] and [b, c] respectively.

It is seen that while fuzzifying crisp matters, we invariably start with the triangular fuzzy number. As a reason as to why it is used, it is always said that it is *easy* to use the triangular fuzzy number. For the uniform probability density function

$$f(x) = 1/ (b - a),\ a \le x \le b,$$

the probability distribution function is given by

$$F(x) = \int_a^x f(x)\, dx = (x - a)/ (b - a).$$

Similarly, for the uniform probability density function

$$g(x) = 1/ (c - b),\ b \le x \le c,$$

the probability distribution function is given by

$$G(x) = (x - b)/ (c - b).$$

It can be seen that F(x) is the left reference function and (1 – G(x)) is the right reference function as defined by Dubois and Prade (see e. g. [1]) of the triangular fuzzy number [a, b, c] with membership

$$\begin{aligned} \mu (x) &= 0, \text{ if } x \le a, \\ &= F(x) = (x - a)/ (b - a), \text{ if } a \le x \le b, \\ &= 1 - G(x) = 1 - (x - b)/ (c - b), \text{ if } b \le x \le c, \\ &= 0, \text{ if } x \ge c. \end{aligned}$$

In fact, just as in the theory of probability the simplest continuous probability law is the uniform probability law, equivalently triangular fuzziness is the simplest possibility law. In other words, what we are trying to assert is found to be satisfactory from this standpoint too. We use triangular fuzzy numbers not just because they are easy to handle. They are in fact the simplest possible normal fuzzy numbers arising out of the simplest possible probability law.

We now state the correct *randomness – fuzziness consistency principle*. Indeed possibility of a value in a given triad [a, b, c] can be expressed as

$$\text{Poss } [X = x] = \theta \text{ Prob } [a \le X \le x] + (1 - \theta) \{1 - \text{Prob } [b \le X \le x]\},$$

where $\theta = 1$, if $a \le x \le b$, and = 0, if $b \le x \le c$. We insist that possibility is expressible either as probability or as a complementary probability. For any individual value of X, possibility is simply a probability of an *event*.

4. The existing probability – possibility consistency principles

We now shift our discussions towards the *probability – possibility consistency principles* which are available in the literature. There are quite a few such principles. In fact, there should not have been more than one such principle had there been a real link between probability and possibility defined on the *same* interval. First, we shall discuss in short regarding use of such consistency principles by various authors. As we shall see, workers in various applicational fields have widely used such probability – possibility consistency principles.

The earliest attempt at making probability and fuzzy set theory work in concert was made by Loginov [12], who interpreted the membership function as a frequentist conditional probability. That attempt was made during the days of the beginning of studies about fuzziness. Zadeh himself [13] later on dismissed that idea, and expressed that probability should be used in concert with fuzzy logic to enhance its effectiveness. In that perspective, probability theory and fuzzy logic were presumed to be complementary rather than competitive.

Before actually constructing the theory of fuzziness, Zadeh [14] felt that the conventional mathematics – the mathematics of precisely defined points, functions, sets, probability measures, etc. were inadequate to deal with certain types of uncertainties. He at that time felt that we need a different kind of mathematics, the mathematics of fuzzy quantities which are not describable in terms of probability distributions. That was his view in 1962. About thirty years later, by 1995, he seemed to have softened his views regarding probability theory as we can see.

Singpurwala and Booker [15] in a survey article discussed about membership functions and probability measures. However, they started speaking in the same language spoken by the workers of fuzzy mathematics. They discussed about imposing probability laws on a given possibility space. They did not try to find whether there could be any mathematical link between the membership functions of a normal fuzzy number and probability distributions.

We conclude that trying to bring about some sort of consistency between a fuzzy membership function and some probability law defined on a fuzzy number is not at all logical, and hence not mathematically meaningful. It has anyway been accepted that imposition of such probability laws is heuristic and arbitrary. We should see that mathematics follows logic. It must not be the other way around in any case. At this point, we would like to refer to a work done by Sheen [16] who has suggested a method of *probabilistic conversion of fuzzy number*, and has gone for conversion of the membership function of a fuzzy number $\mu\ (x)$ into an equivalent probability density function by using one of two linear transformations: proportional probability density function: $p(x) = k_p\ \mu(x)$, and uniform probability density function: $u(x) = \mu(x) + k_u$ where k_p and k_u are values of the conversion constants which ensure that the area under the continuous probability function is equal to 1. When the proportional conversion method is used, the height of the resultant proportional probability density function is independent of the fuzzy number height, but its domain remains the same as that of the original fuzzy number. When the uniform conversion approach is adopted, the domain and the height of the resultant distribution both reduce or increase from their original fuzzy number values. The reduced or increased domain indicates the partial ejection or addition of some members from or to the set. Hence, the uniform distribution reveals certain undesirable properties. Therefore, the application of the proportional density function conversion is recommended in the comparison of fuzzy numbers.

As we can see, Sheen's method of getting proportional probability density function looks like Klir's conversion principle to get probability from possibility. In Sheen's method of conversion, the value of the resultant probability density function is independent of the corresponding fuzzy membership value, while the domain remains the same as that of the original fuzzy number. This is something like Lemaire's conversion of a subnormal fuzzy number to a normal fuzzy number [17]. While Lemaire was absolutely correct in his approach to convert a subnormal fuzzy number to a normal one, Sheen has ended up making everything look very fuzzy indeed. Depending on the value of k_p in $p(x) = k_p\ \mu(x)$ one may end up getting a subnormal fuzzy number also. If Sheen is correct, the subnormal fuzzy number defined by Lemaire should be a probability density function! Obviously, not both Sheen and Lemaire can be correct at the same time. While Lemaire stayed in fuzziness after his conversion from subnormality to normality, Sheen has ended up defining the concept of probability in his *own* way.

We can not define a probabilistic variable following more than one probability laws simultaneously. For example, if a random variable y follows the normal probability law with the probability density function

$$f(y) = \exp\{-(y-\mu)^2/(2\sigma^2)\} / \{\sigma\sqrt{(2\pi)}\}, -\infty < y < \infty,$$

for the same variable we can not say that it follows another probability law defined by another probability density function, for example a triangular probability density, in a different interval of reference. For if such an assumption has to hold good, then we should be able to impose any number of probability laws on the same variable just changing the interval of reference. That would be illogical, and hence outside the philosophy of mathematics.

Further, Sheen's proportional probability density function is unacceptable for one more reason. If we need to construct a probability law followed by a random variable defined in a given interval, there are mathematical rules in the theory of statistical inferences to do so. In fact, statistics too like any other field of knowledge follows certain basic formalisms. Instead, defining first a fuzzy number around a point, and then using some conversion factor to redefine a function obtained from the membership function concerned so that the area under the derived function of reference is equal to 1, and finally calling it a probability density, is totally against the philosophy of statistics. Probability density functions are never constructed in this manner. As for the uniform probability density function defined by Sheen, first, the very nomenclature is incorrect. In the theory of probability, the *uniform probability density* is defined as one that takes a constant value in the interval of reference concerned. At least in mathematics, any standard name of a function must not be used to mean something else. We have discussed in detail matters concerning construction of a normal fuzzy number with the help of two probability laws in one of our recent works [18].

There are three probability – possibility consistency principles, each of them proposed to link probability with fuzzy membership. We would like to stress that had there really been a sort of consistency between probability and possibility defined on the same space, there would have been not three but only one such principle. The very presence of three principles to define just one mathematical formalism is reason enough to suspect that none of them is simply acceptable. As we have shown, to relate probability with possibility, we need two probability spaces to define a possibility space. Therefore the consistency principles in existence are not quite logical. We now proceed to discuss in short the consistency principles available in the literature.

4.1 Zadeh's consistency principle

Zadeh [9] defined the probability-possibility consistency principle stating that a high degree of possibility does not imply a high degree of probability, nor does a low degree of probability imply a low degree of possibility. He defined the degree of consistency between a probability distribution $p = (p_1, p_2, \dots, p_n)$ and the possibility distribution $\pi = (\pi_1, \pi_2, \dots, \pi_n)$ as

$$C_z = \Sigma\, \pi_i\; p_i.$$

Zadeh pointed out that his probability-possibility consistency principle is not a precise law or a relationship between possibility and probability distributions. It is an approximate formalization of the heuristic connection that a lessening of the possibility of an event tends to lessen its probability but not vice-versa. Zadeh himself declared it to be heuristic, and that is an important point to be noted.

Indeed the possibility curve is a culmination of a probability distribution function and a complementary distribution function. As such, if we try to impose one probability law over the entire interval on which the possibility law has been described, we might find that lessening of possibility lessens probability also.

As we have seen, Zadeh tried to define a probability law over the same space over which a possibility law has been defined. Thereafter the scalar product of p and π has been defined as some sort of an index. Here is where we would like to raise a question. Mathematics should follow logic; logic must not be made to follow mathematics or anything else for that matter. But of course, Zadeh himself has commented that the principle is not a precise law. Therefore other such principles have come into picture. We shall now discuss two more such principles available in the literature on fuzzy mathematics.

4.2 Klir's consistency principle

Klir [19] defined a probability – possibility consistency principle as follows. Let $X = \{w_1, w_2, \ldots, w_n\}$ be a finite universe of singletons. Let $p_i = p_i(w_i)$ and $\pi i = \pi_i(w_i)$. The elements of X are assumed to be ordered. Thereafter

$$p_i = \pi_i^{\,1/\alpha} / \Sigma\, \pi_i^{\,1/\alpha}$$

for some parameter α in the open interval (0, 1) defined a two sided relation between π and p.

In effect, defining p_i from π_i in this way to satisfy the uncertainty preservation principle defined by Klir himself is actually nothing but trying to define a probability space, in the measure theoretic sense, from the knowledge of the possibilities concerned. This is nothing but a process of normalizing the values of π_i so that total probability is equal to 1.

It is obvious that probabilities found by normalizing possibilities would of course show consistency in principle. Zadeh's initiative in this regard looked at least better in the sense that in that case probabilities and possibilities were defined independently. In Klir's case, that independence is not there. It looks as though we need a principle of consistency between two things; so define one as a function of the other, and then claim that there is consistency in principle after all. In other words, in this case, the mathematics of normalizing the possibility values has been forced to follow logic. We should see whether we can formalize a set of logic mathematically. Given an event we can consider its probability of happening. Probability can not just be imposed externally on any event. In simple terms, total probability of some related events would be unity; but that does not mean that if some positive fractions sum up to unity, we can impose a probability law suddenly from nowhere. Our point is, either we accept that in a particular situation the theory of fuzzy sets is to be applied, and therefore in that kind of a situation we should set aside theory of probability as a tool unworthy of application in that situation in particular, or we accept that the situation can be handled by applying the theory of probability whence we would not even think of applying the theory of fuzzy sets in that case.

On one hand, if we continue to say that probability theory is inappropriate to deal with a particular situation, while on the other hand we continue to say that we get probabilistic inferences from the membership function concerned, then we would actually be making a philosophical error.

4.3 The Dubois – Prade consistency principle

Dubois and Prade [20] have also put forward another consistency principle. According to them, the possibilistic representation is weaker than the probabilistic one, because it explicitly handles imprecision and because possibility measures are based on ordering structure than an additive one in the probability measures. Thus in going from a probabilistic representation to a possiblistic one, some information is lost because we go from point valued probabilities to interval valued ones; the converse transformation adds information to some possibilistic incomplete knowledge. The transformation from probability to possibility is guided by the principle of maximum specificity, which aims at finding the most informative possibility distribution. While the transformation of possibility to probability is guided by the principle of insufficient reason which aims at finding the probability distribution that contains as much uncertainty as possible but that retains the features of possibility distributions. Dubois and Prade proved that their asymmetric transformation from probability to possibility is the most specific transformation which satisfies the condition of consistency defined by Dubois and Prade themselves in that article.

Here in this case, the reference of the possibility measure has come up. The possibility measure is *not* a measure in the classical sense. We have established that with the help of two probability measures one can study possibility mathematically. We therefore would not like to discuss further about this consistency principle. Defining a set function that does not follow the additivity postulate is one thing, but calling it a *fuzzy* measure is quite another. Instead of calling it a fuzzy measure, if some other name is given, and mathematics proceeds accordingly, no one should have any objection. We insist that fuzziness can definitely be studied measure theoretically in the classical sense.

5. Conclusions

We conclude that:

a. Trying to impose *one* probability space over an interval defining a possibility space is not correct. We need *two*, and not *one*, probability spaces to define a possibility space. On the other hand, given a possibility space, it can be bifurcated into two probability spaces. Hence the measure theoretic matters of possibility must be geared to forming two probability measures.

b. The correct randomness – fuzziness consistency principle should be:

$$\text{Poss}\,[x] = \theta\ \text{Prob}\,[a \le y \le x] + (1 - \theta)\ \{1 - \text{Prob}\,[b \le y \le x]\},$$

where θ is 1, if $a \le x \le b$, and is 0, if $b \le x \le c$. So, possibility of $[X = x]$ for $a \le x \le c$ is expressible as nothing but a probability only, either as Prob $[a \le X \le x]$ or as $\{1 - \text{Prob}\,[b \le X \le x]\}$, whichever is the case. We would like to reiterate that we have used the term probability in the broader measure theoretic sense here.

c. Trying to impose a probability law on the same interval where a normal fuzzy number has been defined, and to try to find a principle of consistency between probability and possibility is not logical. Constructing a probability law from the knowledge of the membership function of a fuzzy number does not make sense. If one is not satisfied with applying probabilistic mathematics in some case, and if it is decided that the mathematics of fuzziness would put forward scope of a better analysis of the situation, then it is better to apply fuzzy mathematical analysis and forget about probability. Applying fuzzy mathematics, and at the same time trying to pull into the picture probability theory in the name of constructing *one* probability law from a possibility law, can not be a correct proposition. We have shown that two probability laws are needed to define a possibility law. If the theory of probability is used to explain fuzziness in this way, then of course it is a different matter. From this standpoint, formalisms of the theory of probability can certainly be applied to explain fuzziness without any heuristic assumptions whatsoever.

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