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A NON-ALGORITHMIC PROOF OF THE FOUR COLOR CONJECTURE

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Abstract

In this article, the Four Color Conjecture has been discussed from a standpoint outside the graph theoretic realm. It has first been shown that in a map with four regions, every region connected to every other, at least one of the regions would be enveloped. With the help of this result, it has been shown that at most three colors are needed to color the boundary regions of a planar map. Finally, it has been proved by induction that four colors are sufficient to color a map, such that no two adjacent regions have the same color. We claim that this is the proper mathematical proof of the four color conjecture, for which the world of mathematics had been waiting for nearly one hundred and sixty years.

Key words: *Planar map, map coloring.*

1. Introduction

Ever since it was stated in 1852, the Four Color Conjecture has been claimed to be mathematically proved quite a few times. Every time the logics were found to be faulty however. Computer dependent algorithmic solutions by Appel and Haken [1], Appel *et al.* [2] and Robertson *et al.* [3] are available. Fritsch and Fritsch [4] discussed about topological as well as combinatorial versions of the problem. Recently, Gonthier [5] has discussed in detail regarding the proof of the conjecture using the concept of reducibility of graphs and related matters.

We are however still calling it a *conjecture* because we still need a proper mathematical proof of the statement on sufficiency of four colors to color a map, such that no two adjacent regions have the same color. Things have meanwhile gone to such a pass that any attempt to prove the conjecture mathematically is greeted skeptically because unfortunately the world of mathematics has meanwhile started to believe that a mathematical proof consisting of arguments based on truths finally arriving at a logical conclusion about this simple conjecture is beyond human capacities. We insist that while remaining within the realm of Graph Theory, a proper mathematical proof of the four color conjecture would never probably be available. Graph theoretic tools can lead to algorithmic proofs only, whether computer dependent or otherwise. In this article, we are going to put forward a proper mathematical proof of the Four Color Conjecture. We have made an attempt to study it from a different perspective. We shall in this article consider the regions of a map directly, instead of using graph theoretic matters such as nodes and edges.

We would like to mention here that our line of proof has got nothing to do with Kempe's work [6]. Kempe tried to show *how* to color a map with four colors. Heawood [7] cited an example that defies Kempe's algorithm. It has been claimed that (see for example [8]) later on Errera [9] has with the help of a counterexample of a graph with seventeen vertices shown that Kempe's algorithm fails in the case of that graph. We are going to establish the result using the method of induction, and this has no connection with Kempe's line of logic.

We would further like to note that the conjecture was about *sufficiency of four colors to color a map*; it was never about *how to do that*. How to color a given map may actually be an algorithmic problem, be that computer dependent or otherwise. The problem of proving the sufficiency of four colors is a different one. These two things must not be superimposed. While remaining within the graph theoretic formalisms, one would always be tempted to find solutions as to *how* to color a map. In other words, the conjecture on sufficiency of four colors to color a map has always been looked into as a problem to *demonstrate* that four colors are sufficient, and unintentionally therefore the problem has been accepted as an algorithmic one since the beginning. In fact, people had tried to find a counterexample to show that the conjecture is wrong, and this process of searching for a counterexample came to a stop only after publication of the Appel-Haken work. What we mean is, right from the start people were trying to find a counterexample defying the conjecture, and therefore were busy in trying to see *how* to color any particular map. No counterexample was ever found, and finally the problem of *how* to color a map with four colors was solved with the help of a computer. In the process, it was simply forgotten that proving the sufficiency of four colors to color a map, and demonstrating how to do that are two totally different things. Indeed, as we have said, even after mathematically proving that four colors are enough to color any map such that no two adjacent regions have the same color, the actual coloration for a map with considerably large number of regions may in fact need the help of a computer. We are interested in finding a mathematical proof, and not really on how to color any given map.

Before discussing our perspective, we would like to add a few comments from eminent mathematicians who had never accepted computer dependent proofs of the conjecture. When D. J. Albers wanted comments from Paul Halmos (see Albers [10], page 19) regarding the work of Appel and Haken and the computer, the following was Halmos' reply: 'I have a religious belief that some day soon, may be 6 months from now, may be 60 years from now, somebody will write a proof of the four color theorem that will take up 60 pages in the *Pacific Journal of Mathematics*. Soon after that, perhaps 6 months or 60 years later, somebody will write a four-page proof, based on concepts that in the meantime we will have developed and studied and understood. The result will belong to the grand, glorious, architectural structure of mathematics.' In other words, Paul Halmos was not at all happy with the computer aided proof.

Computer aided proofs have actually more resonance in computer science than in main-line mathematics [11]. Halmos had in fact a much tougher view regarding the computer aided proofs. We would like quote from Davis and Hersh [12] his objections on such proofs: 'We are still far from having a good proof of the Four Color Theorem. 100 years from now the map theorem will be, I think, an exercise in a first year graduate course, provable in a couple of pages by means of appropriate concepts, which will be completely familiar by then. The present proof relies in effect on an Oracle, and I say, down with Oracle! They are not mathematics.' As we can see, Paul Halmos commented that the computer dependent proof was *not* a good proof.

Tymoczko [13] argued that the Appel–Haken proof is *unsurveyable* because of the enormous calculations involved (see Levin [14]). Tymoczko's argument was simple: if a human being has to do the calculations that were done by the computer in the Appel–Haken proof, it would take an entire lifetime, and probably even that may not be sufficient. There were quite a few philosophical arguments against Tymoczko's views. We are of the view that had there been a small number of possible configurations to deal with, say 3 or 300, then the demonstration would have been surveyable and hence justifiable as a proof. But if the number of configurations to be verified is something like two billion for example as in this case, it is definitely *unsurveyable*, and an unsurveyable demonstration should not be accepted as a formal proof.

After the Appel-Haken-Koch proof, there arrived the Robertson-Sanders-Seymour-Thomas proof. The second proof was better in the sense that it had less computer dependence in comparison to the first. Obviously, people now are expected to have been trying to evolve an algorithm that would not possibly need a computer at all. However, even if an algorithmic yet computer independent proof is found, it would still be long enough. In other words, an algorithmic proof, be that computer dependent or otherwise, would never be a four-page proof that might be ultimately included in a first year graduate course as Paul Halmos wants. Therefore we need a non-algorithmic proof that would be acceptable to all.

We shall in this article first show that in a map with four regions, every region connected to every other, at least one of the regions would always stay *enveloped*, in the sense that a fifth region having a common boundary with this map of four interconnected regions would remain disconnected to the enveloped region or regions. Using this assertion, we would show next that to color the boundary regions of a map with no two adjacent boundary regions having the same color, we need at most three colors. Finally, with the help of these two assertions, we would prove by the method of induction that four colors are indeed sufficient to color a planar map.

2. Our perspective

We first state a text book theorem which states that the graph K_4, with every node connected to every other, is planar (see e.g. [15], page 77). In the following three figures, we are showing the three different types of possibilities in which one can represent the concerned graph K_4 equivalently as a map with four regions A, B, C and D, with every region connected to every other.

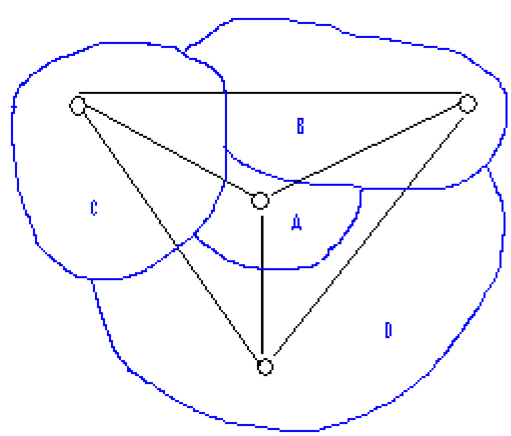

Figure 1. Possibility of type 1: one region enveloped

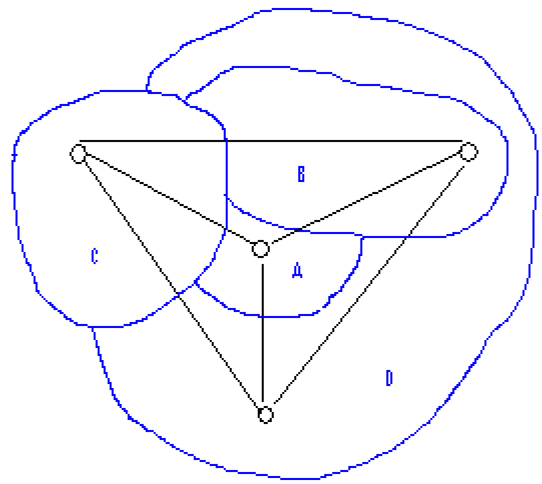

Figure 2. Possibility of type 2: two regions enveloped

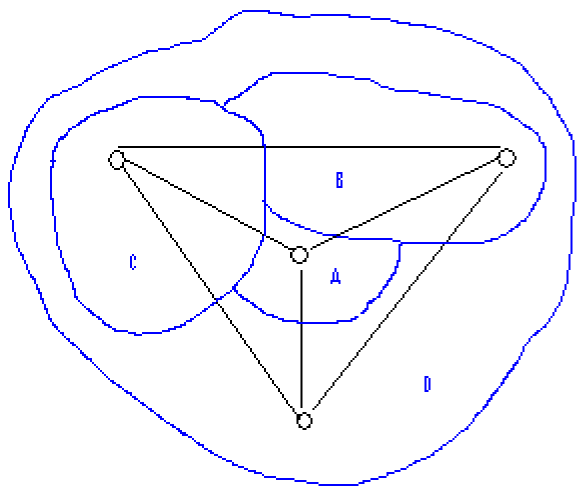

Figure 3. Possibility of type 3: three regions enveloped

In the first type of possibility, *one* of the regions would be *enveloped* by the other three, in the sense that a fifth region representing a fifth node outside our K_4 having some common boundary with one or more of the other three can have no common boundary with the enveloped region . There can be four (= ${}_4C_1$) such cases. In the second type, *two* of the regions would be *enveloped* by the other two. There can be six (= ${}_4C_2$) such cases. In the third type, *three* of the regions would be *enveloped* by the fourth. There can be four (= ${}_4C_3$) such cases. There can be no more cases other the ones enumerated above.

These cases being exhaustive ones, we have in fact established the following theorem:

Theorem - 1: In a planar map with four regions with every region connected to every other, at least one of the regions would be enveloped.

We now proceed to use this theorem to find the how many colors would be sufficient to color a planar map with *five* regions. Theorem – 1 assures us that in a planar map with four regions, every region connected to every other, at least one of the regions would anyway be enveloped. So if a planar map is constituted of five regions, in any case we would not need more than four colors to color that map. In fact, in any given planar map with *five* regions, it is obvious that not all five regions can be connected to one another, because as soon as four of them are connected to one another, the fifth region would have to be disconnected to at least one the interconnected four. Therefore, if even one region remains enveloped, the color used in the enveloped region can be used to color the fifth region. This immediately asserts the following theorem:

Theorem - 2: Four colors are sufficient to color a planar map with five regions.

Taking a planar map as a whole, we can define the totality of the regions which are connected to the complement of the map as its *boundary regions*. We now proceed to find how many colors are needed to color the *boundary regions* of a planar map. *At this point, we are not interested how, and with how many colors, one can color the entire map*.

The rationale behind considering first the boundary regions only is simple. If we consider the union of a given planar map M and its complement M^C as one map again, it is obvious that the boundary regions of M must not need four colors for coloration because in that case we would need a fifth color to color M^C. This will necessitate five colors to color a planar map and that would be against the four color conjecture. In other words, for the planar map M to be four colorable, the boundary regions must necessarily be three colorable.

If three or less regions comprise the boundary regions of the map, the solution is trivial. We now consider the types of cases in which *more than three* boundary regions envelop the other regions of a planar map. If one or more clusters of *three boundary regions* have boundaries common to one another, three colors would be enough. In the absence of that, two colors would sufficient for the boundary regions.

We have earlier proved in Theorem - 1 that *if four regions have boundaries common to one another, at least one of the regions would be enveloped*. Accordingly, there can not in any case be four interconnected regions among the boundary regions of a planar map, for as soon as that would have to happen, at least one of those four *interconnected* regions would be enveloped, and therefore the enveloped region or regions, whichever is the case, would not be in the set of the boundary regions. Hence we would never need four colors to color the boundary regions that envelope the other regions of a planar map. In other words, at most three colors would always be sufficient to color the boundary regions of any planar map. We state this as follows:

Theorem - 3: The boundary regions of a planar map are three colorable.

With the help of the three theorems stated above, we now proceed to prove the Four Color Conjecture. Our perspective is a simple one. We would like to insist once again that the problem is about asserting only that four colors are sufficient for the purpose. How exactly it is to be done in any particular situation is quite another matter. It is true that for coloring a map, an algorithm may actually be needed. We are now going to show *why* four colors are sufficient. *How* four colors can do the job has been demonstrated years ago. Indeed, Theorem-1 stated above is a fact that was overlooked by all earlier workers. This will now lead us to the solution of the problem.

3. The proof of the four color conjecture

We now proceed towards coloration of an entire planar map with n regions. We are aiming to prove by induction that *four colors are sufficient to color a planar map*, *such that no two adjacent regions have the same color*. Let us start with the assumption that *four* colors are sufficient to color a map with number of regions $n = m$. In other words, as the first step towards proving the conjecture by the method of induction, we assume that the statement is true for m regions. We insist that there can not be any reason why such an assumption can not be made. We must not again start questioning *how* the coloration would be done.

We have already established in Theorem - 3 that three colors would be enough for coloring the boundary regions whatever be the situation in the regions other than those on the boundary. Hence if we add one more region on the boundary, *four* colors would be sufficient to color *the resultant boundary region* of the map with number of regions $n = (m + 1)$.

In other words, if it is assumed that four colors are sufficient to color a map with number of regions $n = m$, it can bee seen that the statement holds for an extension of the concerned map with number of regions $n = (m + 1)$.

Consider now a map with *five* regions. We are considering a map with five regions because with the number of regions four or less, the conjecture is trivially true. Theorem – 2 assures us regarding sufficiency of four colors to color a map for $n = 5$.

We have already shown that if it is assumed that four colors are sufficient to color a map with number of regions $n = m$, it can bee seen that the statement holds for an extension of the concerned map with number of regions $n = (m + 1)$. Hence it must be true for $n = 6$, and so on.

This establishes by induction that four colors are sufficient to color a planar map with no two adjacent regions having the same color.

We have thus proved that the Four Color Conjecture is indeed true. Only now are we in a position to declare that it is a *theorem* on its own right and not a conjecture any more.

4. Conclusions and discussions

Thus far no algorithm-independent proof of the four color conjecture could be found because the theory of graphs would probably lead to algorithm dependent proofs only. So as to establish the conjecture without the help of algorithms, one way would be to step outside the graph theoretic formalisms. We have found that going back to the root, if the map coloring problem is attacked right as coloring the different regions of the map, it does actually lead to a proof. Basically, we have to note that in a map of four regions, if the regions are connected to one another, at least one of the regions would have to be disconnected to a fifth region having some common boundary with the boundary of the map comprising of the four given regions. This finally leads to a proof of the four color conjecture. Some people still might have a tendency to ask how exactly the coloration would have to be done. That is a different problem, an algorithmic one, and solutions of that problem have already been found many years ago. We insist that the computer dependent solutions are demonstrations only. They are not really proofs in the classical sense. In fact, even if a computer- independent algorithmic solution is found out using concepts of spirals and such other related concepts, that would still not be a classroom proof in the strictly classical sense.

We have noted further that though it looks obvious that in the planar graph K_4 only one node is not connectable to a fifth node outside the graph keeping planarity intact, if we look into the matters from our perspective not just one region but upto a maximum of three regions can remain disconnected to a fifth region outside the map with the four given regions, each of the four regions being connected to one another. This should give us an entirely new perspective of looking into certain unsolved graph theoretic conjectures. We believe that using our approach, one may arrive at other interesting results that are not quite obvious if looked into graph theoretically. Our proof depends on the simple assertion that five regions can never be interconnected to one another, and this was something no one has earlier looked into. Based on this simple fact, the four color theorem thus could be seen to be true.

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This work has been dedicated in honor of Professor Paul Richard Halmos who never accepted the computer dependent proofs of the four color theorem as formal proofs.