Mortal Ammensal and an Enemy Ecological Model with Immigration for Ammensal Species at a Constant Rate

K.V.L.N.Acharyulu
Faculty of Science,
Department of Mathematics
Bapatla Engineering College,
Bapatla-522101,India
kvlna@yahoo.com

N.Ch. Pattabhi Ramacharyulu
Professor (Retd.) of Mathematics
Department of Mathematics & Humanities
National Institute of Technology
Warangal – 506004, India
patabhi1933@yahoo.com

Abstract

The paper throws light on the investigation of the global stability of a Mathematical model of Mortal Ammensal and an Enemy Ecological Model with Immigration for Ammensal Species at a Constant Rate. It is instituted by Liapunov's stability criteria by constructing a suitable Liapunov's function for appraising the global stability of the model in the case of co-existence equilibrium state.

Key words: *Equilibrium states, Stability, Liapunov's function for global stability*

1. Introduction

Mathematical ecology explains how diverse kinds of plants and animals can live together in the same area for generations together sharing common resources. Such a sharing may continue till one or other or some of the species go extinct locally. However there are instances of several species of different nature coexisting persistently in the same habitat in spite of inevitable limitations on space and resources. As such, ecology may also be referred as the study of distribution and abundance of species in the same habitat availing the same resources. Ecology relates to the study of living beings (animals and plants) in relation to their habits and habitats. This discipline of knowledge is a branch of Evolutionary Biology purported to explain how or to what extent the living beings are regulated in nature. Mathematical Modeling in Mathematical Biosciences is an attempt to identify and describe some instances of our routine in the language of Mathematics. It is an endeavor as old as the first human being and as modern as tomorrow's newspaper. While widening and deepening the scope of Mathematical Modeling in life and medical sciences, one is not just restricted to the use of mathematical techniques, which are already known.

Mathematics, the queen of Sciences is yet to be employed more effectively as a tool in understanding several phenomena in Life Sciences when compared to the extent it is put to frequent use in Physical and Engineering Sciences. As diverse models in any branch of Science and Technology, Mathematical Models in theoretical ecology are also of great importance and utility because they answer and raise questions related to natural phenomena.

K.V.L.N.Acharyulu and N.ch.Pattabhi Ramacharyulu [1-6] studied the Local stability of ecological Ammensalism with miscellaneous dimensions. Local stability analysis for an Ammensal- enemy eco-system with various resources in different cases has been fulfilled in the author's earlier work. The present investigation is mainly concentrated on the establishment of the global stability of the co-existent equilibrium state of a Mathematical model of Mortal Ammensal and an Enemy Ecological Model with Immigration for Ammensal Species at a Constant Rate by utilizing a property constructed by Liapunov's function with Liapunov's criteria for global stability.

1.1. Liapunov's Stability Analysis

A.M. Liapunov invented a meritable method in 1892 to test the global stability of equilibrium points in the case of linear and non-linear systems. This method affords stability information directly without solving the differential equations involved in the system. Hence it is also called Liapunov's direct method to observe the criteria for global stability. His method is based on the main charactestic of constructing a scalar function called Liapunov's function. The stability behaviour of solutions of linear and weak non-linear system is done by using the techniques of variation of constants formulae and integral inequalities. So this analysis is confined to a small neighbourhood of operating point i.e., local stability. Further, the techniques used there in require explicit knowledge of solutions of corresponding linear systems. Hence, the stability behaviour of a physical system is curbed by these limitations. In this connection Several autors like Lotka[10], Kapur[7,8],Srinivas [11] and Lakiminarayan[9] etc.applied this method in various situations for global stability.

1.2. Liapunov's Method for Global stability

Consider an autonomous system

$$\frac{dx}{dt} = F_1(x, y) \text{ and } \frac{dy}{dt} = F_2(x, y) \tag{1}$$

Assume that this system has an isolated initial point taken as (0, 0). Consider a function E(x,y) possessing continuous partial derivatives along with the path of (1).This path is represented by C= [(x (t), y (t)] in the parametric form. E(x,y) can be regarded as a function of 't' along C with rate of change

If the total energy of physical system has a local minimum at a certain equilibrium point then the point is said to be stable .Liapunov generalized this principle by constructing a function $E(N_1, N_2)$ whose rate of change is given by

$$\frac{\partial E}{\partial t} = \frac{\partial E}{\partial N_1}.\frac{\partial N_1}{\partial t} + \frac{\partial E}{\partial N_2}.\frac{\partial N_2}{\partial t} = \frac{\partial E}{\partial N_1}F_1 + \frac{\partial E}{\partial N_2}F_2 \qquad (2)$$

corresponding to the system.

(ii)Theorem (A): If there exists a Liapunov's function E (x,y) for the system (1), then the critical point (0,0) is stable. Further, if this function has additional property that the function (2) is negative definite, then the critical point (0, 0) is asymptotically stable. (3)

The following theorem provides scope to ascertain the definiteness of a Liapunov's function.

(iii)Theorem (B): The function $E(x,y) = ax^2+bxy+cy^2$ is positive definite if $a>0$ and $b^2-4ac<0$ and negative definite if $a<0$, $b^2-4ac<0$. (4)

1.3. Notation Adopted

N_1 and N2 are the populations of the Ammensal and enemy species with natural growth rates a_1 and a_2 respectively.

a_{11} is rate of decrease of the Ammensal due to insufficient food.

a_{22} is rate of decrease of the enemy due to insufficient food.

a_{12} is rate of increase of the Ammensal due to inhibition by the enemy.

$h_1= a_{11}H_1$ is rate of harvest of the Ammenssal spieces.

$K_i = a_i/a_{ii}$ are the carrying capacities of N_i, i = 1, 2

$\alpha = a_{12}/a_{11}$ is the coefficient of Ammensalism.

The state variables N_1 and N_2 as well as the model parameters a_1, a_2, a_{11}, a_{22}, K_1, K_2, α, h_1, h_2 are assumed to be non-negative constants.

2. Basic Equations

The model equations for the Ammensalism between two species with the decay Ammensal is being harvested (immigration) at a constant rate are given by the following system of non-linear first ordered ordinary differential equations.

(i). Equation for the growth rate of the Ammensal species $\left(S_1\right)$:

$$\frac{dN_1}{dt} = a_{11}\left[-K_1N_1 - N_1^2 - \alpha N_1N_2 + H_1\right] \tag{5}$$

(ii). Equation for the growth rate of the enemy species $\left(S_2\right)$:

$$\frac{dN_2}{dt} = a_{22}N_2\left[K_2 - N_2\right] \tag{6}$$

3. Equilibrium States

The system has the following two equilibrium points given by $\frac{dN_i}{dt} = 0,\ i = 1,2$.

(i) $$\overline{N_1} = \frac{\sqrt{K_1^2 + 4H_1} - K_1}{2} \quad ; \quad \overline{N_2} = 0 \quad \text{(Enemy- washed out State)} \tag{7}$$

(ii) $$\overline{N_1} = \frac{\sqrt{\left(K_1 + \alpha K_2\right)^2 + 4H_1} - \left(K_1 + \alpha K_2\right)}{2} \quad ; \quad \overline{N_2} = K_2 \quad \text{(Co-existent State)} \tag{8}$$

4. Global Stability analysis for the model

The global stability of this model with the system (5 and (6) is tested. It is already observed that this system is stable at non-trivial co-existent equilibrium point. i.e

$$\overline{N_1} = \frac{\sqrt{\left(K_1 + \alpha K_2\right)^2 + 4H_1} - \left(K_1 + \alpha K_2\right)}{2} \quad ; \quad \overline{N_2} = K_2$$

The linearized perturbed equations over the perturbations (u_1, u_2) are

$$\frac{dU_1}{dt} = -2a_{11}\left[\overline{N_1} + \left(\frac{K_1 + \alpha K_2}{2}\right)\right]U_1 - \alpha a_{11}\overline{N_1}U_2 \tag{9}$$

$$\frac{dU_2}{dt} = -a_{22}\overline{N_2}U_2 \tag{10}$$

The corresponding characteristic equation is

$$\left(\lambda + 2a_{11}\left[\overline{N_1} + \left(\frac{K_1 + \alpha K_2}{2}\right)\right]\right)\left(\lambda + a_{22}\overline{N_2}\right) = 0 \tag{11}$$

$$\lambda^2 + \left[2a_{11}\left[\overline{N_1} + \left(\frac{K_1 + \alpha K_2}{2}\right)\right] + a_{22}\overline{N_2}\right]\lambda + 2a_{11}a_{22}\left[\overline{N_1} + \left(\frac{K_1 + \alpha K_2}{2}\right)\right]\overline{N_2} = 0 \tag{12}$$

Equation (12) is of the form $\lambda^2 + P\lambda + Q = 0$

where
$$p = 2a_{11}\left[\overline{N_1} + \left(\frac{K_1 + \alpha K_2}{2}\right)\right] + a_{22}\overline{N_2} > 0 \tag{13}$$

$$q = 2a_{11}a_{22}\left[\overline{N_1} + \left(\frac{K_1 + \alpha K_2}{2}\right)\right]\overline{N_2} > 0 \tag{14}$$

$\therefore$ The conditions for the existence of Liapunov's function are satisfied.

Now, we define

$$\text{E}\,(U_1, U_2) = \frac{1}{2}\left(AU_1^2 + 2BU_1U_2 + CU_2^2\right) \tag{15}$$

where

$$A = \frac{\left(a_{22}\overline{N_2}\right)^2 + 2a_{11}a_{22}\overline{N_2}\left[\overline{N_1} + \left(\frac{K_1 + \alpha K_2}{2}\right)\right]}{D} \tag{16}$$

$$B = -\frac{\alpha a_{11}a_{22}\overline{N_1}\,\overline{N_2}}{D} \tag{17}$$

$$C = \frac{4a_{11}^2\left[\overline{N_1} + \left(\frac{K_1 + \alpha K_2}{2}\right)\right]^2 + \left(\alpha a_{11}\overline{N_1}\right)^2 + 2a_{11}a_{22}\left[\overline{N_1} + \left(\frac{K_1 + \alpha K_2}{2}\right)\right]\overline{N_2}}{D} \tag{18}$$

And $\quad D = PQ > 0 \qquad (19)$

From the equations (13) and (14) it is clear that D>0 and A>0.

Also

$$D^2\left(AC - B^2\right) = D^2\left\{\left(\frac{\left(a_{22}\overline{N_2}\right)^2 + 2a_{11}a_{22}\left[\overline{N_1} + \left(\frac{K_1 + \alpha K_2}{2}\right)\right]\overline{N_2}}{D}\right)\left(\frac{4a_{11}^2\left[\overline{N_1} + \left(\frac{K_1 + \alpha K_2}{2}\right)\right]^2 + \left(\alpha a_{11}\overline{N_1}\right)^2 + 2a_{11}a_{22}\left[\overline{N_1} + \left(\frac{K_1 + \alpha K_2}{2}\right)\right]\overline{N_2}}{D}\right) - \frac{\left(\alpha a_{11}a_{22}\overline{N_1}\,\overline{N_2}\right)^2}{D^2}\right\}$$

$$\Rightarrow D^2\left(AC - B^2\right) > 0 \Rightarrow AC - B^2 > 0 \qquad \text{i.e., } B^2\text{-AC} < 0 \tag{20}$$

$\therefore$ The function E (U_1, U_2) at (15) is positive definite.

Further

$$\frac{\partial E}{\partial U_1}\frac{dU_1}{dt}+\frac{\partial E}{\partial U_2}\frac{dU_2}{dt}=\left[\left(AU_1+BU_2\right)\left(-2a_{11}\overline{N_1}+\left(\frac{K_1+\alpha K_2}{2}\right)U_1-\alpha a_{11}\overline{N_1}U_2\right)+\left(BU_1+CU_2\right)\left(-a_{22}\overline{N_2}U_2\right)\right] \tag{21}$$

Substituting the values of A, B and C from (16) (17) and (18) in (21) we get

$$\frac{\partial E}{\partial U_1}\frac{dU_1}{dt}+\frac{\partial E}{\partial U_2}\frac{dU_2}{dt}=-\frac{1}{D}\left[2a_{11}\left(\overline{N_1}+\left(\frac{K_1+\alpha K_2}{2}\right)+a_{22}\overline{N_2}\right)\left(2a_{11}a_{22}\overline{N_2}\left(\overline{N_1}+\left(\frac{K_1+\alpha K_2}{2}\right)\right)\right)\right]U_1^2$$

$$+\frac{1}{D}\left[-\alpha a_{11}a_{22}{}^2\overline{N_1}\,\overline{N_2}{}^2-2\alpha a_{11}{}^2a_{22}\overline{N_1}\,\overline{N_2}\left(\overline{N_1}+\left(\frac{K_1+\alpha K_2}{2}\right)\right)+2\alpha a_{11}{}^2a_{22}\overline{N_1}\,\overline{N_2}\left(\overline{N_1}+\left(\frac{K_1+\alpha K_2}{2}\right)\right)+\alpha a_{11}a_{22}{}^2\overline{N_1}\,\overline{N_2}{}^2\right]U_1U_2$$

$$-\frac{1}{D}\left[\left(2a_{11}a_{22}\overline{N_2}\left(\overline{N_1}+\left(\frac{K_1+\alpha K_2}{2}\right)\right)\right)\left(a_{22}\overline{N_2}+2a_{11}\left(\overline{N_1}+\left(\frac{K_1+\alpha K_2}{2}\right)\right)\right)\right]U_2^2$$

$$\frac{\partial E}{\partial U_1}\frac{dU_1}{dt}+\frac{\partial E}{\partial U_2}\frac{dU_2}{dt}= -\frac{1}{D}\left[DU_1^2+DU_2^2\right] \tag{22}$$

$$= -\left(U_1^2+U_2^2\right) \tag{23}$$

$$\therefore\ \frac{\partial E}{\partial U_1}\frac{dU_1}{dt}+\frac{\partial E}{\partial U_2}\frac{dU_2}{dt}=-\left(U_1^2+U_2^2\right) \tag{24}$$

which is clearly negative definite. So, E(U_1,U_2) is a Liapunov's function for the linear system.

Next we prove that E(U_1,U_2) is also a Liapunov's function for the non-linear system.

Let f_1 and f_2 be two functions of N_1 and N_2 defined by

$$f_1(N_1,N_2)= a_{11}\left(-K_1N_1-N_1^2-\alpha N_1N_2+H_1\right) \tag{25}$$

$$f_2(N_1,N_2)= a_{22}\ N_2\left[K_2-N_2\right] \tag{26}$$

Now we have to show that $\frac{\partial E}{\partial U_1}f_1+\frac{\partial E}{\partial U_2}f_2$ is negative definite.

By putting $N_1=\overline{N_1}+U_1$; $N_2=\overline{N_2}+U_2$ in (5) and (6), we get

$$f_1(U_1,U_2)=\frac{du_1}{dt}= a_{11}\ \left(-K_1\left(\overline{N_1}+U_1\right)-\left(\overline{N_1}+U_1\right)^2-\alpha\left(\overline{N_1}+U_1\right)\left(\overline{N_2}+U_2\right)+H_1\right)$$

$$= -2a_{11}\overline{N_1}+\left(\frac{K_1+\alpha K_2}{2}\right)U_1-\alpha a_{11}\overline{N_1}U_2-a_{11}U_1^2-\alpha a_{11}U_1U_2$$

$$\Rightarrow \quad f_1\left(U_1,U_2\right) = \frac{dU_1}{dt} = -2a_{11}\overline{N_1} + \left(\frac{K_1+\alpha K_2}{2}\right)u_1 - \alpha a_{11}\overline{N_1}U_2 + F\left(U_1,U_2\right) \tag{27}$$

$$\text{where } F\left(U_1,U_2\right) = -a_{11}U_1^2 - \alpha a_{11}U_1U_2 \tag{28}$$

Similarly

$$f_2\left(U_1,U_2\right) = \frac{du_2}{dt} = a_{22}\ (\overline{N_2}+U_2)\left[K_2-(\overline{N_2}+U_2)\right]$$

$$= a_{22}K_2\overline{N_2} - a_{22}\overline{N_2}^2 - a_{22}\overline{N_2}U_2 + a_{22}k_2U_2 - a_{22}\overline{N_2}U_2 - a_{22}U_2^2$$

$$= -a_{22}\overline{N_2}U_2 + a_{22}U_2\left(K_2-\overline{N_2}\right) - a_{22}U_2^2$$

$$\Rightarrow f_2\left(U_1,U_2\right) = \frac{dU_2}{dt} = -a_{22}\overline{N_2}U_2 + G\left(U_1,U_2\right) \tag{29}$$

$$\text{where } G\left(U_1,U_2\right) = -a_{22}U_2^2 \tag{30}$$

From (15)

$$\frac{\partial E}{\partial U_1} = AU_1 + BU_2 \tag{31}$$

$$\frac{\partial E}{\partial U_2} = BU_1 + CU_2 \tag{32}$$

Now

$$\frac{\partial E}{\partial U_1}f_1 + \frac{\partial E}{\partial U_2}f_2 = \left(AU_1+BU_2\right)\left(-2a_{11}\overline{N_1} + \left(\frac{K_1+\alpha K_2}{2}\right)U_1 - \alpha a_{11}\overline{N_1}U_2 + F\left(U_1,U_2\right)\right)$$

$$+ \left(BU_1+CU_2\right)\left(-a_{22}\overline{N_2}U_2 + G\left(U_1,U_2\right)\right) \tag{33}$$

$$\frac{\partial E}{\partial U_1}f_1 + \frac{\partial E}{\partial U_2}f_2 = -\left(U_1^2+U_2^2\right) + \left(AU_1+BU_2\right)F\left(U_1,U_2\right) + \left(BU_1+CU_2\right)G\left(U_1,U_2\right) \tag{34}$$

Introducing polar co-ordinates $U_1 = r\cos\theta$, $U_2 = r\sin\theta$, the equation (34) can be written as

$$\frac{\partial E}{\partial U_1}f_1 + \frac{\partial E}{\partial U_2}f_2 = -r^2 + r[(A\cos\theta + B\sin\theta)F\left(U_1,U_2\right) + (B\cos\theta + C\sin\theta)\,G\left(U_1,U_2\right)] \tag{35}$$

Let us denote the largest of the numbers $|A|$, $|B|$ and $|C|$ by L.

Our assumptions imply that $\left|F\left(U_1,U_2\right)\right| < \frac{r}{6L}$ and $\left|G\left(U_1,U_2\right)\right| < \frac{r}{6L}$ for all sufficiently small r > 0.

So, $$\frac{\partial E}{\partial U_1}f_1 + \frac{\partial E}{\partial U_2}f_2 < -r^2 + \frac{4Lr^2}{6L} = -\frac{r^2}{3} < 0 \tag{36}$$

Thus E(U_1, U_2) is a positive definite function with the condition that $\frac{\partial E}{\partial U_1} f_1 + \frac{\partial E}{\partial U_2} f_2$ is negative definite.

$\therefore$ The equilibrium state E_2 is **"asymptotically stable"** globally.

5. Conclusion

The global stability analysis of a mathematical model of Mortal Ammensal and an Enemy Ecological Model with Immigration for Ammensal Species at a Constant Rate is fulfilled and it is noticed that this model is stable at co-existent state.

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Authors

K.V.L.N.Acharyulu: He is working as a Lecturer in the Department of Mathematics, Bapatla Engineering College, Bapatla which is a prestigious institution of Andhra Pradesh. He took his M.Phil. Degree in Mathematics from the University of Madras and stood first in R.K.M. Vivekananda College,Chennai. Nearly for the last Ten years he is rendering his services to the students and he is applauded by one and all for his best way of teaching. He has participated in some seminars and presented his papers on various topics. More than 31 papers were published in International Journals. He has submitted his thesis for the award of Ph.D under the able guidance of Prof. N.Ch.Pattabhi Ramacharyulu.

N.Ch.Pattabhi Ramacharyulu: He is a retired professor in the Department of Mathematics & Humanities, National Institute of Technology, Warangal. He is a stalwart of Mathematics. His yeoman services as a Lecturer, Professor, Professor Emeritus and Deputy Director enriched the knowledge of thousands of students. He has nearly 35 PhDs and plenty number of M.Phils to his credit. His International and National research papers more than 200 were published in various esteemed national and international journals. He is a Member of Various Professional Bodies. He published four books on Mathematics. He received so many prestigious awards and rewards. He is a walking Encyclopedia on various subjects of modern Mathematics.