The Application of Fractional Brownian Motion in Option Pricing

Qing-xin Zhou

School of Basic Science,Harbin University of Commerce,Harbin
zhouqingxin1981@126.com

Abstract

In this text, Fractional Brown Motion theory during random process is applied to research the option pricing problem.

Firstly, Fractional Brown Motion theory and actuarial pricing method of option are utilized to derive Black-Scholes formula under Fractional Brown Motion and form corresponding mathematical model to describe option pricing.

Secondly, based on BYD stock, estimation model on volatility of this stock is given to analyze and calculate the stock price volatility.

Finally, make instance analysis for BYD's option. Based on market data of BYD's stock and option, calculate the actual option price and theoretical price of BYD by Black-Scholes formula under Fractional Brown Motion. Compare the forecast price of this stock option given by model with actual price, relatively good effect is obtained, and then conclude that the model has relatively strong applicability.

Keyword*s: Stock Prices; Black-Scholes Model; Brown Movement; Fractional Brownian Motion; Options; Volatility*

1. Introduction

Brown Motion theory is usually used for researching the change of asset price in previous option pricing theory, while comparatively speaking, Fractional Brown Motion theory has related nature between incremental quantities, and using it to research asset price can more reflect some characteristics of stock yield with more extensive significance. In this text, Fractional Brown Motion theory during random process is applied to research option pricing problem. [1-4].

Modern option pricing theory began in 1990. The French mathematician Louis Bachelie first put forward option pricing model, and this theory thinks that stock price should be subject to Brown Motion, which is considered as the naissance mark of asset pricing theory.

In 1973, on the premise of no-arbitrage principle, Fisher Black and Myron Scholes derived partial differential equations of European style option pricing under a series of ideal assumptions that stock price should be subject to logarithmic normal distribution and so on, that is famous Black-Scholes formula. Almost at the same time, Robert Merton improved Black-Scholes model to make the model more practical.

In 1976, J.C.Cox and S.A.Ross put forward risk neutral pricing theory.

In 1979, J.M.HarriSon and D.Krep put forward to describe no-arbitrage market and incomplete market by martingale method, which had a profound effect on future development of financial mathematics.

In 1988, Mogens Blat and Tina Hviid Ryberg derived Black-Scholes pricing formula of European style option by only using little actuarial knowledge, which is the beginning of actuarial pricing method applied in option pricing[5-7].

Nowadays, many domestic scholars utilize Black-Scholes formula under Fractional Brown Motion to research option problem.

2. Basic Theory

Definition1[8] If random process $\{X(t), t \geq 0\}$ satisfies：

(1) $X(0) = 0$;

(2) $\{X(t), t \geq 0\}$ is stationary independent increment;

(3) For every $t > 0$, it has $X(t) \Box N(0, c^2 t)$,

then random process $\{X(t), t \geq 0\}$ is Wiener Process or Brown Motion.

Definition 2 [8] Suppose $\{B(t), t \geq 0\}$ is standard Brown Motion

$$X(t) = e^{\mu t + \sigma B(t)}, \qquad t \geq 0$$

is called Geometry Brown Motion with obedience drift parameter μ, fluctuation parameter σ.

Definition 3 [9,10] Suppose (Ω, F, P) denotes a probability space, H is a constant between $(0,1)$（also be called Hurst Parameter）, if random process $B_H(t)$ satisfies

(1) $B(0) = 0$;

(2) For random $t \in R^+$, $B_H(t)$ is a random variable, and makes $E\{B_H(t)\} = 0$;

(3) For random $t, s \in R^+$, it has

$$E\{B_H(t)B_H(s)\} = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$

then call random process $\{B_H(t)\}$ as Fractional Brown Motion.

Lemma 1 The most common form of Fractional Brown Motion is

$$B_H(t) = \frac{1}{\Gamma(H + \frac{1}{2})} \int_0^t (t-\tau)^{H-\frac{1}{2}} w(\tau) d\tau$$

where $\Gamma(x)$ is Γ function, $w(\tau)$ is Gauss White Noise with the mean of 0

property 1: The increment of Fractional Brown Motion is:

$$\Delta B_H(t,s) = B_H(t) - B_H(s),$$

Property2：

$$E(\Delta B_H(t,s)) = E(B_H(t) - B_H(s))$$
$$= E(B_H(t)) - E(B_H(s))$$
$$= 0 \qquad \forall t, s \in R$$

It can be seen that $E(\Delta B_H(t,s))$ Has nothing to do with the current moment.

property3：

$$E((\Delta B_H(t,s))^2) = E((B_H(t) - B_H(s))(B_H(s) - B_H(0)))$$
$$= E(B_H(t)^2) + E(B_H(s)^2) - 2E(B_H(t)B_H(s))$$
$$= t^{2H} + s^{2H} - 2 \cdot \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)$$

$$= \left| t - s \right|^{2H}$$

Property4： $E(\Delta B_H(t,s)\Delta B_H(s,0))$

$$= E((B_H(t) - B_H(s))(B_H(s) - B_H(0)))$$

$$= E(B_H(t)B_H(s)) - E(B_H(t)B_H(0)) - E(B_H(s)^2) + E(B_H(s)B_H(0))$$

$$= \frac{1}{2}(\left| t \right|^{2H} - \left| s \right|^{2H} - \left| t - s \right|^{2H})$$

When $H = \frac{1}{2}$, Fractional Brown Motion is Brown Motion.

Obviously, Brown Motions are mutual independence at different times, but Fractional Brown Motion has more continuity, so it has more universal applicability to use Fractional Brown Motion to explain real natural phenomenon.

Lemma 1[11] Suppose risk-free interest rate is r, expected yield of underlying assets price is μ , its price is $S(t)$ at time t ,then expiration time is T , the price $V(t)$ of European style call option of exercise price X at time $t(\ t < T)$ is

$$V(t) = E((e^{-\mu(T-t)}S(T) - e^{-r(T-t)}X)I(e^{-\mu(T-t)}S(T) > e^{-r(T-t)}X))$$

Lemma 2 [11] Suppose expiration time is T ,Exercise Price is X , The initial value for the stock price is $S(0)$,and the price satisfies the equation

$$dS(t) = \mu S(t)dt + \sigma dB_H(t),$$

where μ is expected yield of underlying assets price, σ is Stock volatility, and random process $\{B_H(t)\}$ is Fractional Brown Motion, then the price of European style call option at time $t(\ t < T)$ is

$$V^* = S(t)N(d_1^*) - e^{-r(T-t)}XN(d_2^*) \tag{1}$$

where

$$d_1^* = \frac{\ln\dfrac{S(t)}{X} + r(T-t) + \dfrac{1}{2}\sigma^2(T^{2H} - t^{2H})}{\sigma\sqrt{T^{2H} - t^{2H}}}$$

and

$$d_2^* = \frac{\ln\dfrac{S(t)}{X} + r(T-t) - \dfrac{1}{2}\sigma^2(T^{2H} - t^{2H})}{\sigma\sqrt{T^{2H} - t^{2H}}}$$

Formula (1) is Black-Scholes formula under Fractional Brown Motion and, in this text, the option pricing formula defined by formula (1) is called Model I.

Definition 4 European style pricing formula of call option about continuous time ---
--Black-Scholes formula

The Black-Scholes formula is as follows:

$$C(S,T,K) = S(0)\Phi(\xi) - KR^{-T}\Phi(\xi - \sigma\sqrt{T})$$

Where

$$\xi = \frac{\ln(SR^T/K)}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$$

Usually, time is the single moment $R = r + 1$, but for practical problems, moments are near continuous, then $R = e^r$, so the formula becomes:

$$C(S,T,K) = S(0)\Phi(\xi) - Ke^{-rT}\Phi(\xi - \sigma\sqrt{T})$$

where

$$\xi = \frac{\ln(S(0)e^{rT}/K)}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}.$$

3. Application of Model I in BYD Option Pricing

Volatility is also called mutability, which is the estimation for risk degree of stock market. The stock price volatility is a kind of measurement for uncertainty of stock price trend, and the magnitude of stock price volatility can affect option value. In the option pricing formula of ex dividend payment, the main five input variables are basic stock price, exercise price of option, expiration time of option, risk-free interest rate and volatility of stock price. The first three input variables can be obtained by option contract clause, risk-free interest rate can also be represented by general deposit and loan interest rate or treasury bond rate, and critical input variable related to successful use of Black-Scholes option pricing formula is volatility [12, 13].

3.1. Method of Using History Data to Estimate Stock Price Volatility

Stock price is subject to logarithmic normal distribution [14], that is:

$$\ln S(T) - \ln S(t) \square \varphi\left[\left(\mu - \frac{\sigma^2}{2}\right)(T-t), \sigma\sqrt{T-t}\right]$$

Among which $S(T)$ is the stock price at future time T; $S(t)$ is the stock price at present time t; μ is expected increase rate of stock price; σ is stock price volatility; $\varphi(a,b)$ means normal distribution of mean value a, standard deviation b.

Using history data to estimate stock index volatility, and time interval of observation index is fixed.

The definition is as follows: n is n time interval, S_i is stock index at end of No. i time interval, $i = 1, 2, \cdots$. τ takes year as unit, and expresses the length of time interval. Make $u_i = ln(S_i/S_{i-1})$, due to $S_i = S_{i-1}\exp u_i$, so u_i is continuous compounding yield within No. i time interval. The usual estimated value of standard deviation of u_i is

$$S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \overline{u})^2}$$

where $\overline{u}$ in above formula is average value of u_i.

Known from above discussion, the standard deviation of u_i is $\sigma\sqrt{\tau}$, so the variable S is estimated value of $\sigma\sqrt{\tau}$, σ can be estimated as S', then

$$\sigma = S' = \frac{S}{\sqrt{\tau}}$$

The standard error of this estimation approximates $S'/\sqrt{2n}$.

When using history data to estimate volatility, the following situations should be explained:

(1) The choice of value n is very important. As a general rule, if it has more data, the result is more accurate. But in reality, the value σ is time-varying, and very old history data has no effect on predicting future, so latest 82 days closing indexes are chosen as data material. The volatility at certain time that will be estimated and materials that will be adopted should stay the same in the aspect of time.

(2) When estimating and using volatility index, experience has shown that transaction days should be adopted, and market closing days of exchange should be deducted when calculating volatility [14].

3.2. Estimation of BYD Stock Market Volatility

In this text, the closing price indexes of BYD transaction days from January 6,2014 to May 9, 2014 are taken as research object. The data on first day is closing price of January 6, 2014, number of time interval is 81, observation times are $n + 1 = 82$, and length of time interval expressed by unit of day is the reciprocal of time interval number contained in this day.
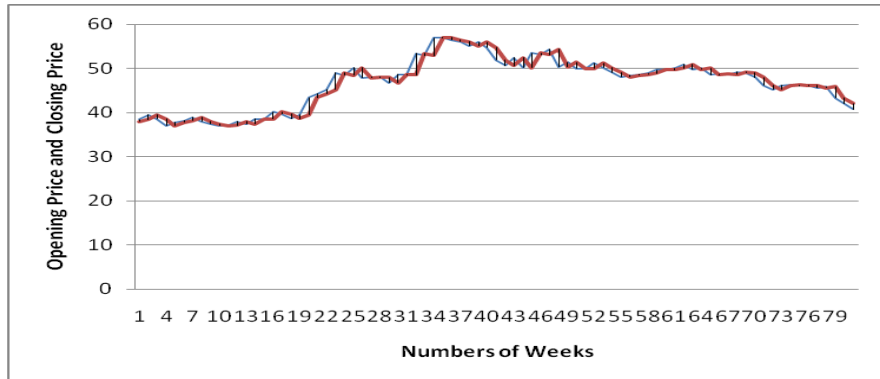

Figure 1. BYD Stock Information from January 6,2014 to May 9, 2014

Calculate by history data

$$\sum_{i=1}^{81} u_i = 0.069624251$$

and

$$\sum_{i=1}^{81} u_i^2 = 0.078607609$$

The estimated value of standard deviation of daily return rate is

$$S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n} u_i^2 - \frac{1}{n(n-1)}(\sum_{i=1}^{n} u_i)^2} = 0.031334134$$

The estimated value of volatility σ is

$$s' = \frac{S}{\sqrt{\tau}} \times 100\% = 28.0261014\%$$

Its standard deviation is

$$\frac{s'}{\sqrt{2n}} \times 100\% = 2.202\%$$

Then its stock volatility of BYD estimated from above data is 28.026% and its standard deviation is 2.202%.

3.3. Application of Model I in BYD Option

The below is empirical analysis of BYD option price of Model I.

Firstly, estimating volatility of above section can obtain $\sigma = 0.280261014$. The basic information of BYD option is shown in Table 1 specifically.

Table 1. Option Contract of BYD Option

Item	Content
Abbreviation More	BYD
Stock Code	002594
Exercise Proportion	1
Exercise Price	2.6
Settlement Method	Delivery Settlement
Transaction Period	From January 6, 2014 to May 9, 2014
Expiration Date	May 17, 2014

Table 2.Comparison of Actual Price and Theoretical Price of BYD Option

Time	Option Actual Price	Option Theoretical Price
2014-1-6	11.35	13.65
2014-1-7	11.8	14.1
2014-1-8	11.92	14.22
2014-2-7	9.94	15.14
2014-2-10	10.05	15.25
2014-2-11	9.73	14.93
2014-3-3	11.3	9.7
2014-3-4	11.47	13.07
2014-3-5	11.61	13.21
2014-4-11	12.17	8.37
2014-4-14	12.12	15.92
2014-4-15	12	15.8
2014-5-5	11.37	13.67
2014-5-6	11.05	11.25
2014-5-7	10.89	11.09
2014-5-8	10.72	10.02
2014-5-9	10.71	10.81

From above data: BYD option transacted from January 6, 2014 to May 9, 2014, price change is relatively smooth during this period, while the fluctuation of transaction theoretical value calculated according to fractional Black- Scholes model is relatively large. Draw Figure 1 with Matlab and make comparison, seeing details in Figure 2.
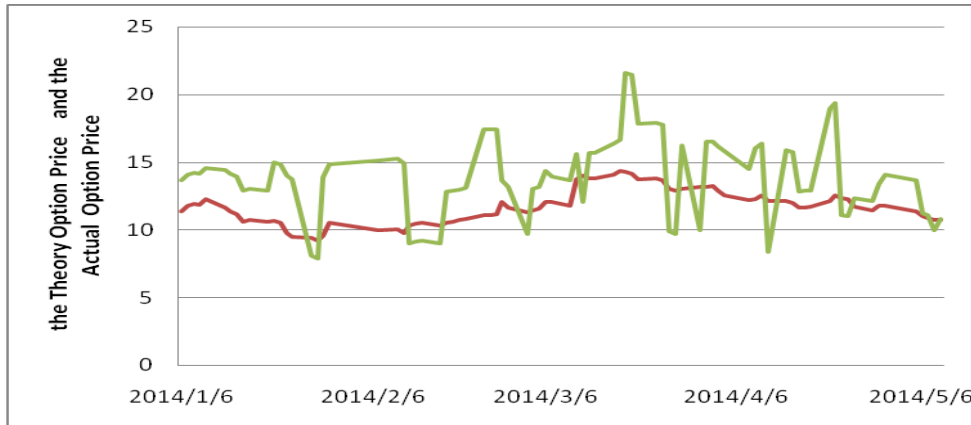
Figure 2. Trend Chart of Option Actual Prices and Option Theoretical Prices of BYD Option

Among which the volatility index is 0.28026; the intraday closing price of BYD is taken as stock price; the exercise price index is 2.6 Yuan; risk-free return rate is 15.36% (above data is from Great Wisdom); and Suppose $H = 0.8$

3.4. Result Analysis of Model I

From line graph, it can be obviously seen that the price change of BYD option is relatively smooth during this period, while the transaction theoretical value calculated according to fractional Black- Scholes model is relatively fluctuating, but surround the actual value to change. The overall price difference is not big. The difference between actual value and theoretical value is very small in front section, but as time grows, the predicted effect is worse and worse, so predicted option price can not explain actual change trend of option, that is to say, the change of option market is unpredictable.

Analyze difference existence reason of BYD actual option price and theoretical option price, I think that it can be divided into the following points:

(1) The select of value H is not accurate enough, and value H should be a changeable value all the time, for convenience, in this text, only one numerical value is selected as representation, so compared to actual value, a certain deviation is caused.

(2) Stock volatility σ is not real numerical value, it is predicted by model, so difference is caused between theoretical value and actual value;

(3) The select of start data can also cause corresponding deviation on general prediction;

From above reasons, it can be fully explained the difference existence reasons of BYD actual option price and theoretical option price.

According to some reasons described above, I think the following adjustments can make the theoretical value and actual value being more identical:

(1) Selecting value H by subsection for many times, and different applicable value H is chosen for every section of data to carry out prediction and calculation of model;

(2) The prediction model with best stock volatility σ is selected to carry out prediction and select more sample data as far as possible to make the prediction more accurate;

(3) Selecting a new start data again to carry out model prediction for every some data to make the theoretical value and actual value being more identical;

(4) Try to select one stock which more conforms to model condition to research or choose some time of one stock which conform to model condition to research, analyze and predict.

Through some solving strategies which put forward above, the problems researched in this text can be solved better and try to avoid deviation.

4. Conclusion

In this paper, a stochastic process Brown motions and sports scores construct a mathematical model describes the different pricing options.

Pricing for solving the problem of risky assets and its derivatives is one of the elements of the study of mathematical finance, and option pricing problem is one of the most important content. With the continuous development and improvement of the stock market, more and more people use some new mathematical tools to analyze and study of option pricing. Previous theories commonly used to study the Brownian Motion changes in asset prices , and sports scores relatively Brown with incremental inter -related nature, Using it to study some of the characteristics of asset prices which better reflect the stock returns, and there is a broader sense. This paper is to study the nature of the option pricing theory of stochastic processes.

The main findings are as follows:

The first part is introduction, introducing the research background and its significance.

The second part describes the Brown campaign, geometric Brown, Brown definitions and nature of sports score.

The third part is the application of this article, based on the model of BYD options example analysis, we used the model to predict the price of the stock options and compared with actual prices.

Seen from Figure 3, the effect is relatively considerable. The general value difference is not big, but volatility difference is relatively large, although difference of this method exists through several experiments, it just reflects that stock market is changeable and unpredictable.

Acknowledgement

References

[1]   K. Wang, "Foreign Exchange Option Pricing Model and Empirical Research Based on Fractional Brown Motion", Chongqing University, **(2008)**.

[2]   C. J. He, "Research on Option Pricing Model in Fractional Brown Motion Environment", Hefei University of Technology, **(2009)**.

[3]   J. C. Feng, "Research on Some Problems of Option Pricing under Fractional Brown Motion Environment", Lanzhou University, **(2009)**.

[4]   X. T. Lu, "Option Pricing Problem under Fractional Brown Motion Environment", Huazhong University of Science and Technology, **(2008)**.

[5]   J. Wang and J. Wang, "Application on Random Process Theory in Pricing Option", Beijing Jiaotong University, **(2007)**.

[6]   S. L. Li, "Research on Option Pricing Problem under Fractional Brown Motion", Harbin Engineering University, **(2012)**.

[7]   L. Zhang, "Application and Research on Actuarial Studies in Option Pricing", Shaanxi Normal University, **(2008)**.

[8]   N. Wiener, "The Average of an Analytic Functional and the Brownian Movement", Norbert Wiener: Collected Works with Commentaries, MIT Press, vol. 1, **(1976)**.

[9]   N. Aronszajn, "Theory of Reproducing Kernels", Trans. Amer. Math. Soc., **(1960)**, pp. 337-404.

[10] G. S. Ladde and L. Wu, "Development of modified Geometric Brownian Motion models by using stock price data and basic statistics", Nonlinear Analysis, **(2009)**, vol. 71, no. 12, pp.1203-1208.

[11] L. Bachelier, "Theory of Speculation", Cooter P. The random Character of Stock Market Prices, Cambridge: MIT Press, **(1900)**, pp.17-78.

[12] F. Black, M. Scholes, "The Pricing of Option and Corporate Liabilities", Journal of Political Economy, **(1973)**, pp. 81, 637-659.

[13] P. A. Samuelson, "Rational Theory of Warrant Price", Industrial Management Review, **(1965)**, pp. 6, 13-31.

[14] X. Gabaix, X. Gopik, P. Rishnan, V. Plerou, "A Theory of Power Laws in Financial Fluctuations", **(2003)**, pp. 423, 267-270.